II — Principal Axes of a Euclidean Cloud

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This text is adapted from the chapter 2 of the monograph
Multiple Correspondence Analysis (QASS series n°163, SAGE, 2010)

II.1 Basic geometric notions

Elements of a geometric space: *points*, *line*, *plane*.

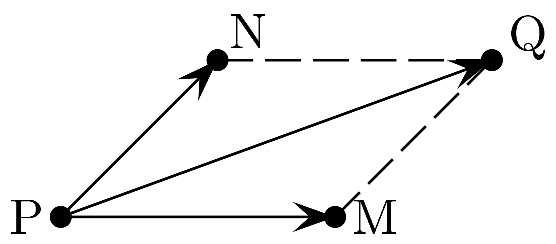
— *Affine notions*: alignment, direction and barycenter.

Couple of points (P, M), ou *dipole* $\longrightarrow$ *vecteur* $\overrightarrow{\mathrm{PM}}$

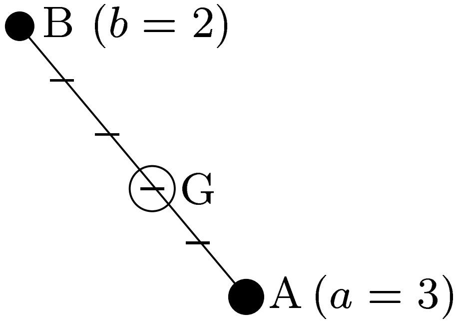
• The deviation from point P to point M is M − P (“terminal minus initial”), that is, $\overrightarrow{\mathrm{PM}}$.

Deviations add up vectorially: sum of vectors by *parallelogram rule*

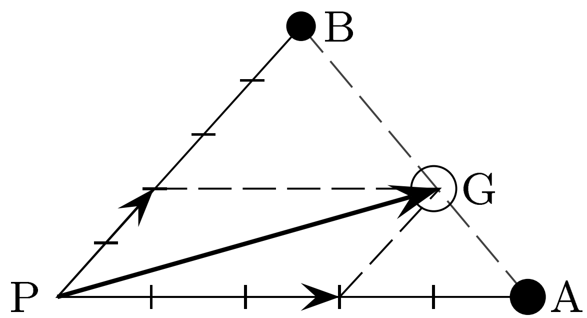
$$\overrightarrow{\mathrm{PM}} + \overrightarrow{\mathrm{PN}} = \overrightarrow{\mathrm{PQ}}$$



- Barycenter of a dipole



$G = \frac{3A+2B}{5}$

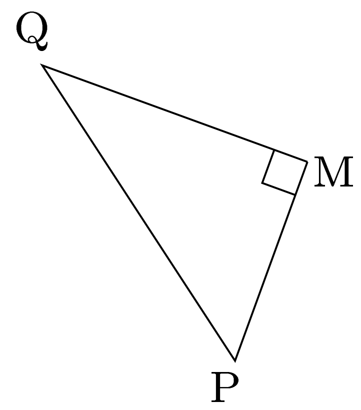


$\overrightarrow{PG} = \frac{3}{5}\,\overrightarrow{PA} + \frac{2}{5}\,\overrightarrow{PB}$

Barycenter = *weighted average of points*: $G = \frac{aA+bB}{a+b}$

— *Metric notions*: distances and angles.

Triangle inequality: $\text{PQ} \leq \text{PM} + \text{MQ}$



Pythagorean theorem:
If $\overrightarrow{\text{PM}}$ and $\overrightarrow{\text{MQ}}$ are perpendicular then:

$$(\text{PM})^2 + (\text{MQ})^2 = (\text{PQ})^2$$

(triangle MPQ with right angle at M),

II.2 Cloud of Points

Target example

Figure 1. Target example (10 points)

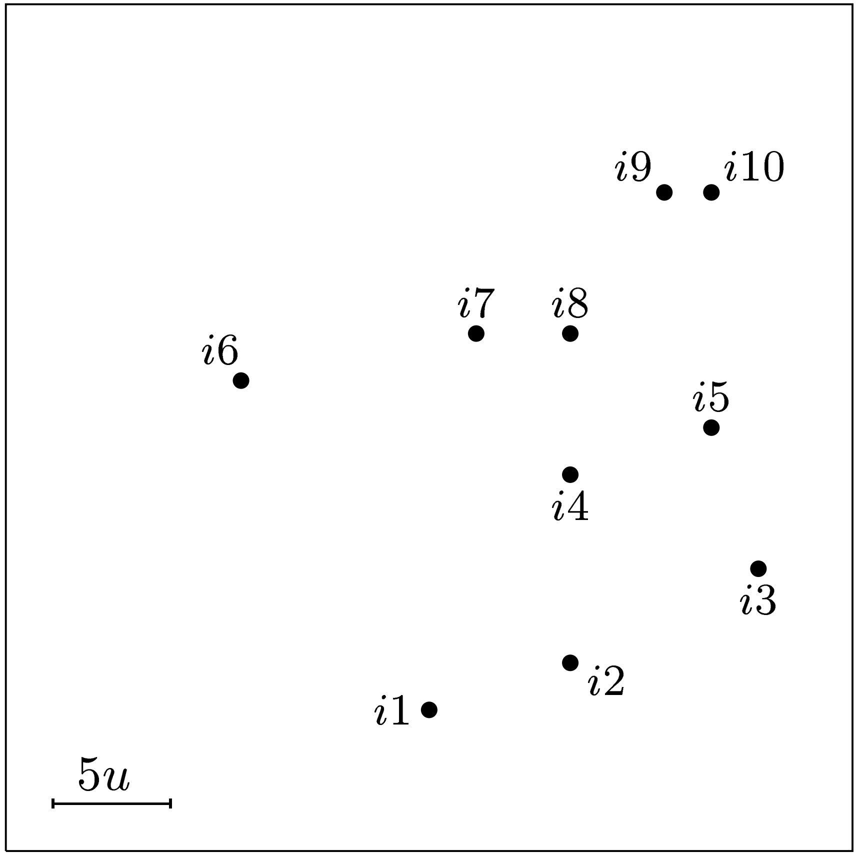
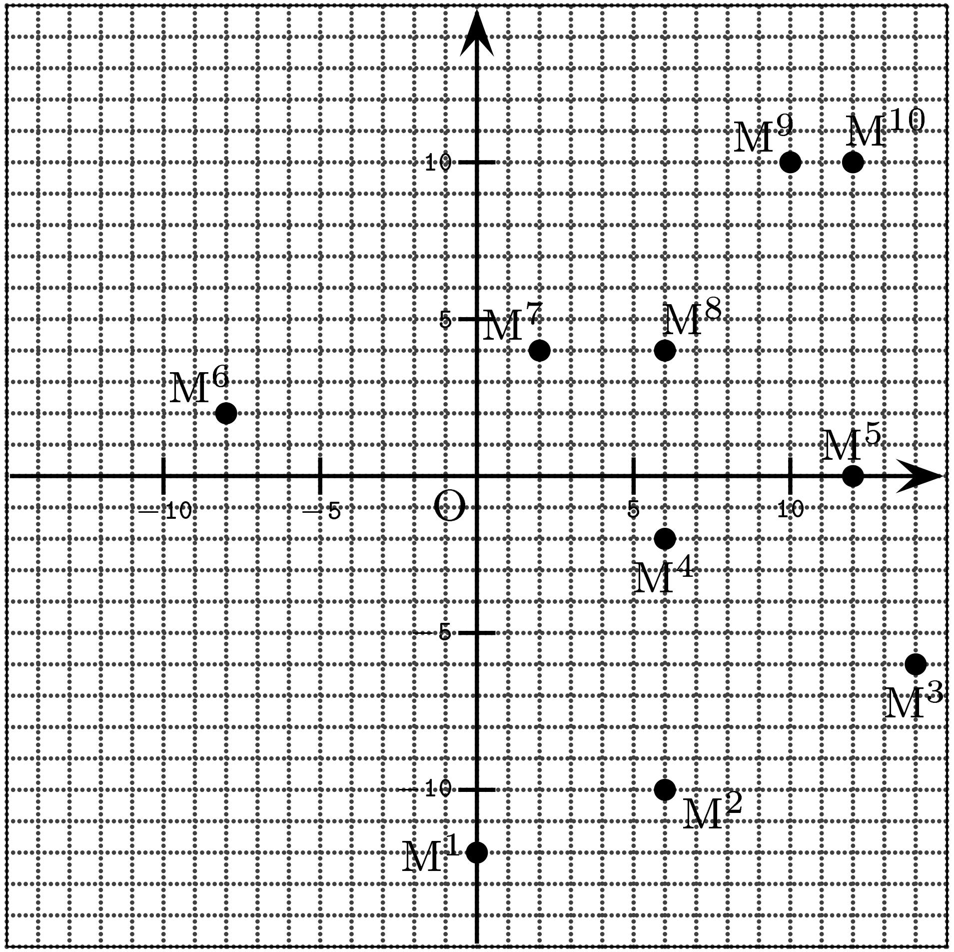


Figure 1bis. Cloud of 10 points with origine-point O and initial axes

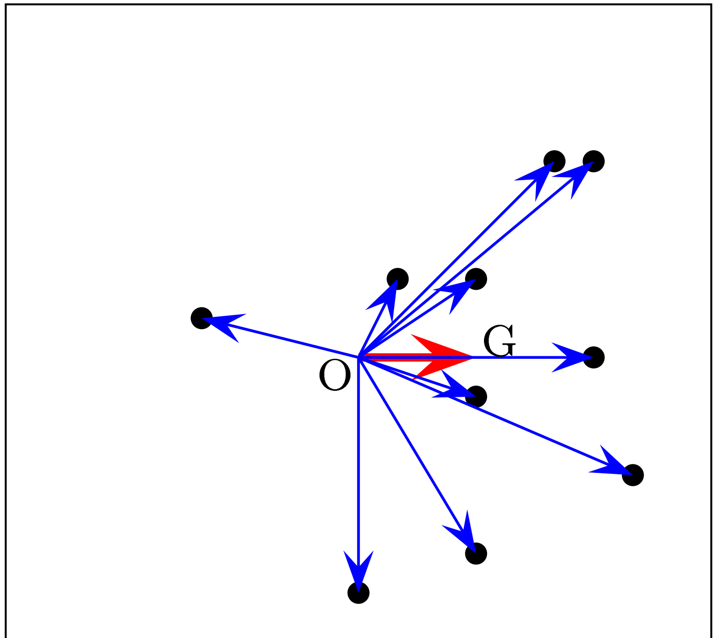


Initial coordinates

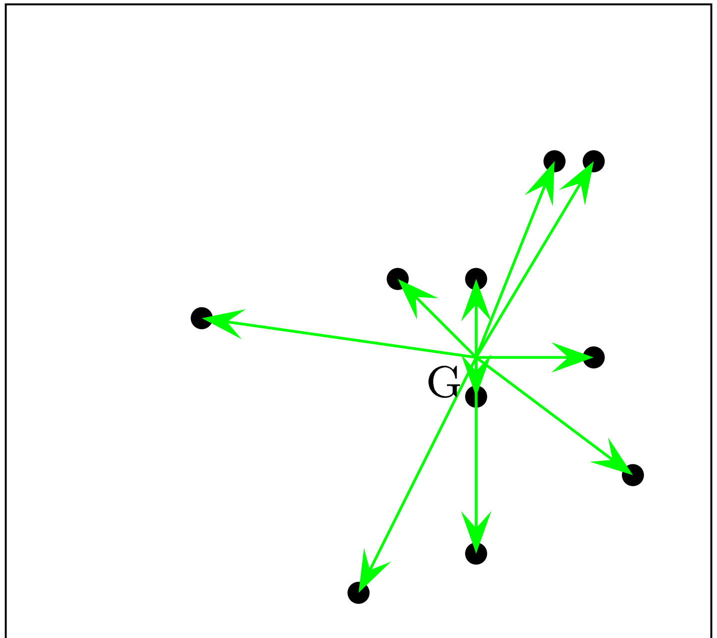
	x_1	x_2
M^1	0	−12
M^2	6	−10
M^3	14	−6
M^4	6	−2
M^5	12	0
M^6	−8	2
M^7	2	4
M^8	6	4
M^9	10	10
M^{10}	12	10
Means	6	0
Variances	40	52
Covariance	+ 8	

Mean point: point G

$$\overrightarrow{\mathrm{OG}} = \sum p_i \, \overrightarrow{\mathrm{OM}}^i \qquad \sum p_i \, \overrightarrow{\mathrm{GM}}^i = \overrightarrow{0} \text{ (barycentric property).}$$



$\overrightarrow{\mathrm{OG}} = \frac{1}{n} \sum \overrightarrow{\mathrm{OM}}^i$



$\sum \overrightarrow{\mathrm{GM}}^i = \overrightarrow{0}$

Target Example: $p_i = \dfrac{1}{n}$

Variance of a cloud :

$$V_{\text{nuage}} = \sum p_i \, (\text{GM}^i)^2$$

(see Benzécri 1992, p.93)

Property. In rectangular axes, the variance of the cloud is the sum of the variances of the coordinate variables.

Contribution of point M^i:

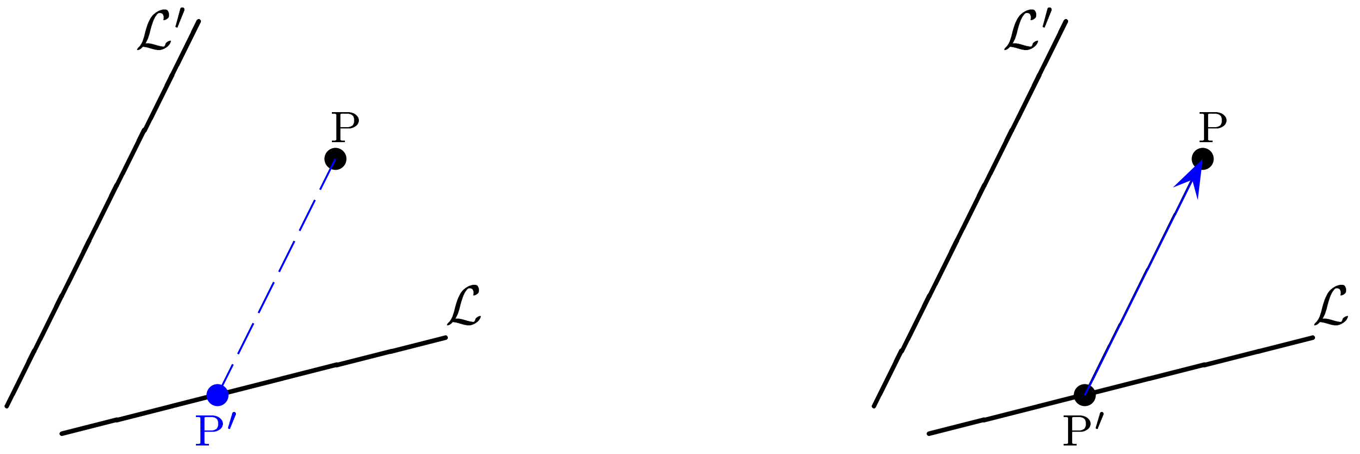
$$\text{Ctr}_i = \frac{p_i (\text{GM}^i)^2}{V_{\text{nuage}}}$$

II.3 Principal Axes of a Cloud

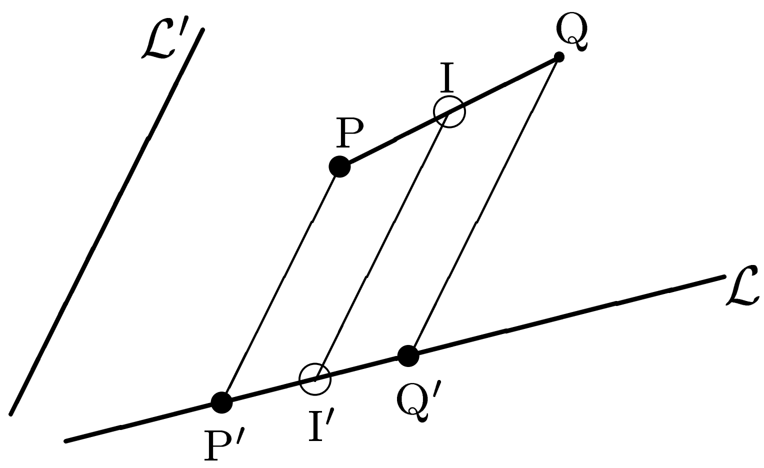
Projection of a cloud

$\mathrm{P}' =$ projection of point P onto $\mathcal{L}$ along $\mathcal{L}'$

$\overrightarrow{\mathrm{P}'\mathrm{P}} =$ residual deviation

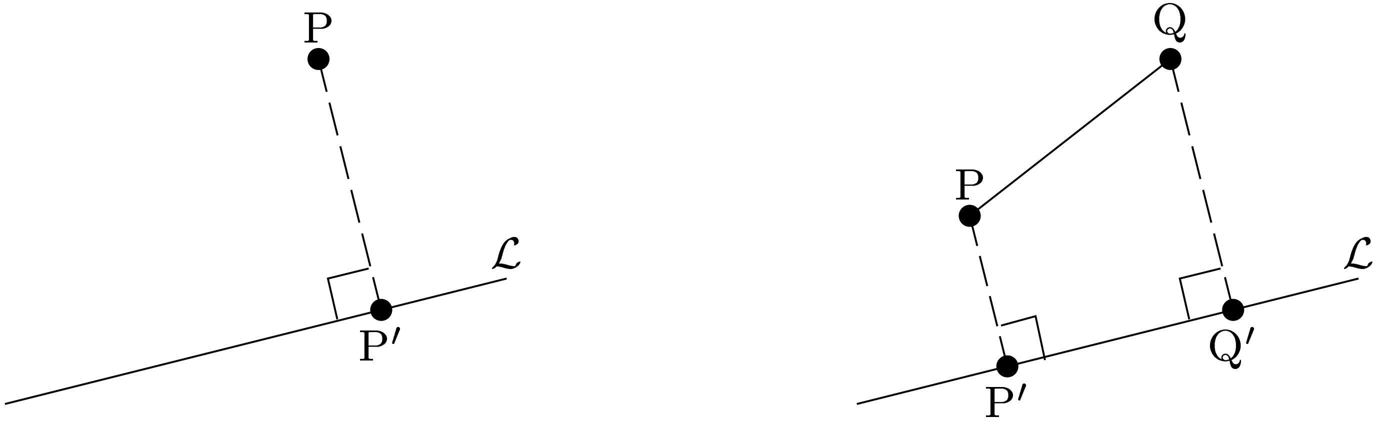


If I is the midpoint of P and Q, the projection I′ of I on $\mathcal{L}$ is the midpoint of P′ and Q′.



Mean point property: The mean point is preserved by projection

Orthogonal projection: PP$'$ is perpendicular to $\mathcal{L}$.



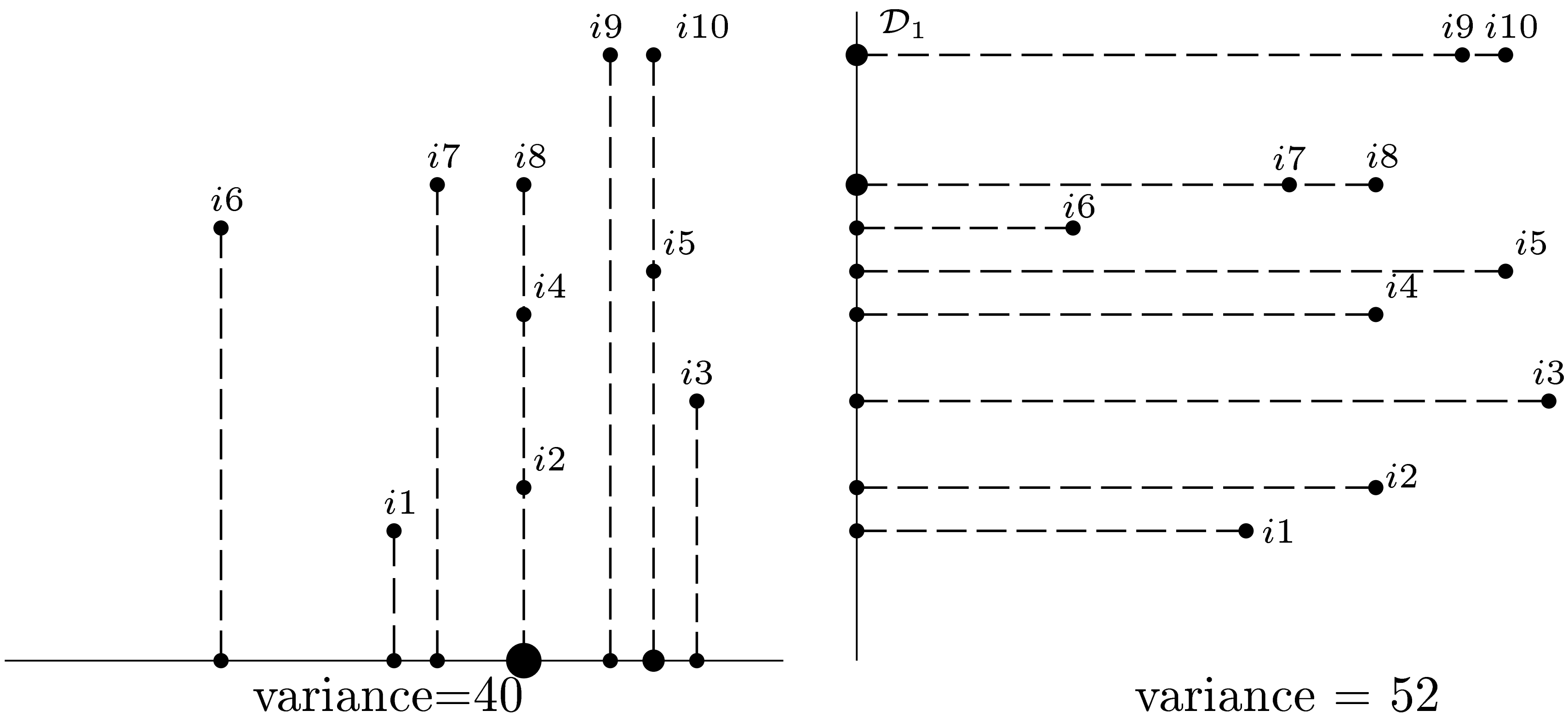
Property: the orthogonal projection contracts distances

$$\mathrm{P'Q'} \leq \mathrm{PQ}$$

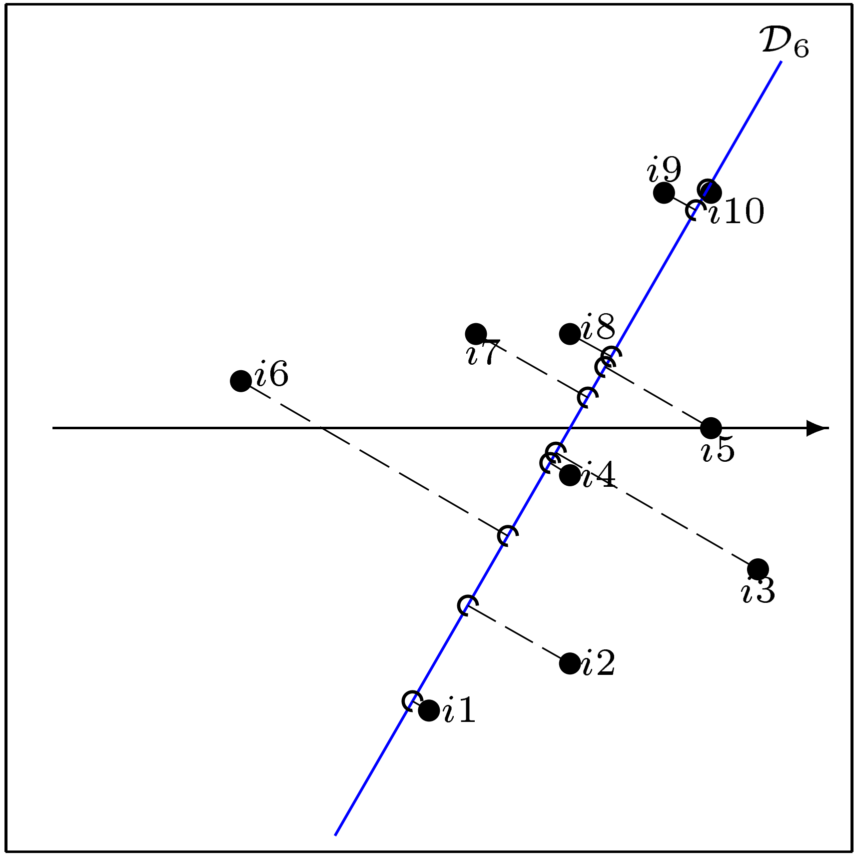
therefore

variance of a projected cloud $\leq$ variance of the initial cloud.

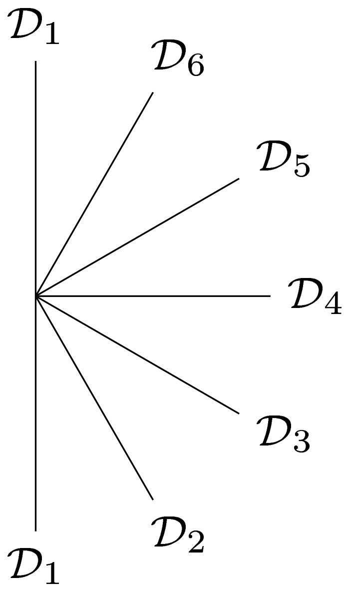
Projected clouds on several lines

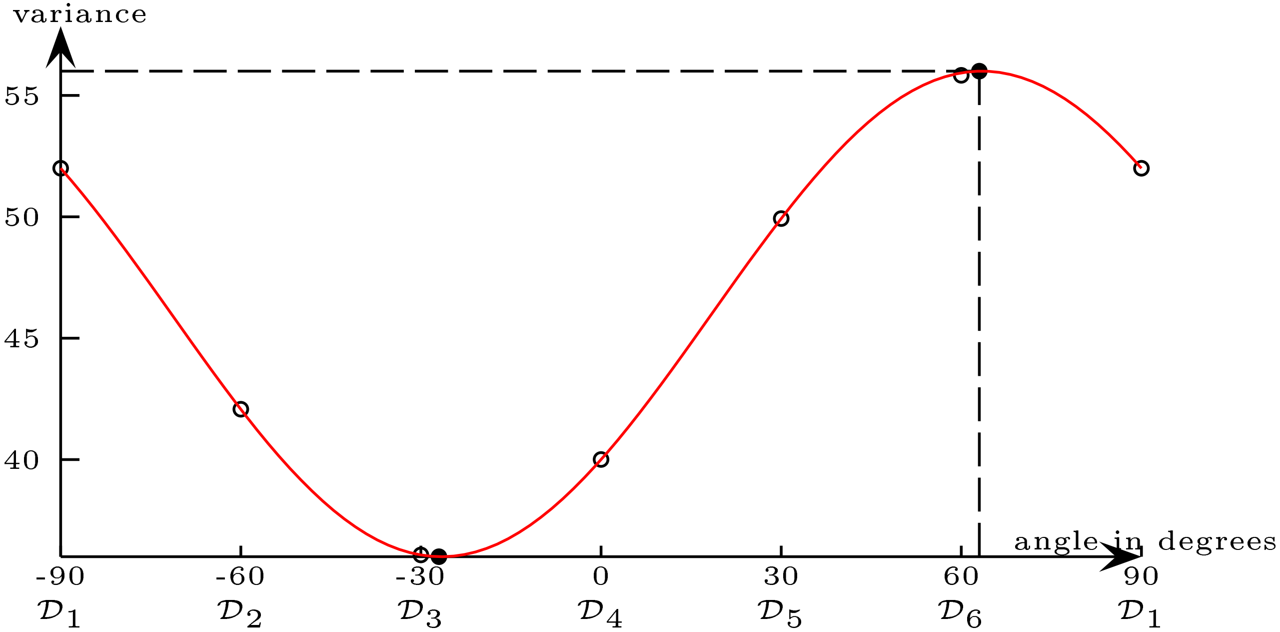


Orthogonal additive decomposition : the sum of the variances of projected clouds onto perpendicular lines is the variance of initial cloud: $40 + 52 = 92$.



Projection onto an oblique line (60 degrees) : variance = 55.9

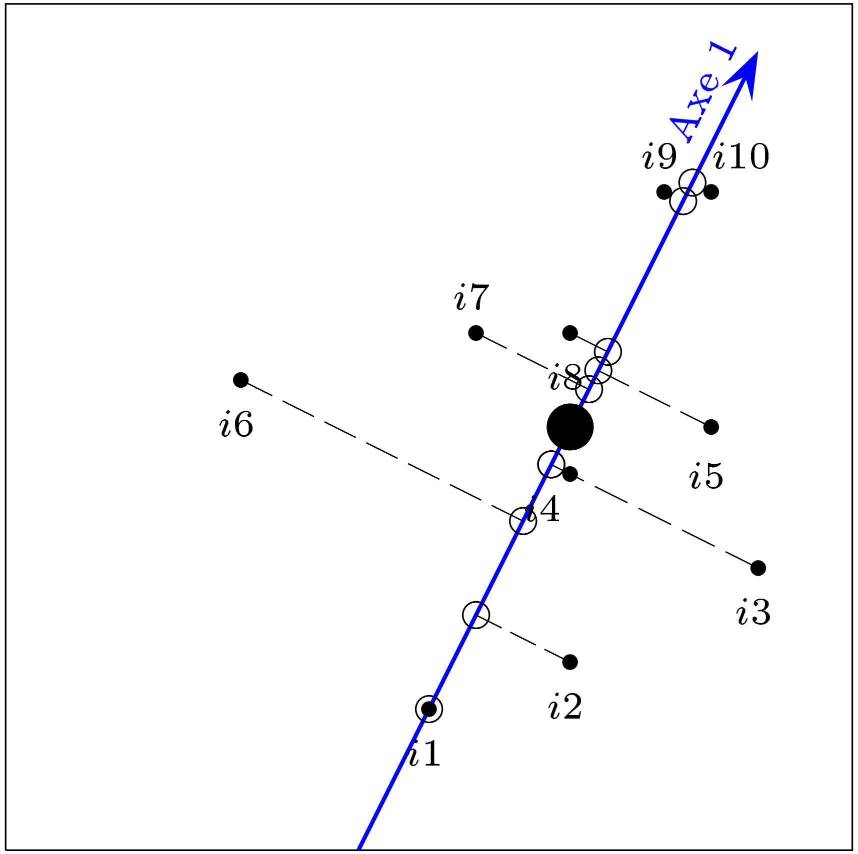


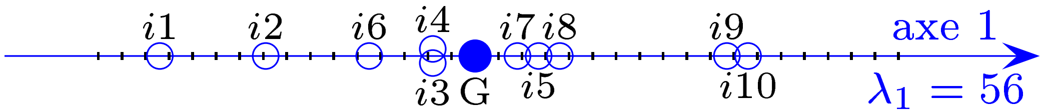


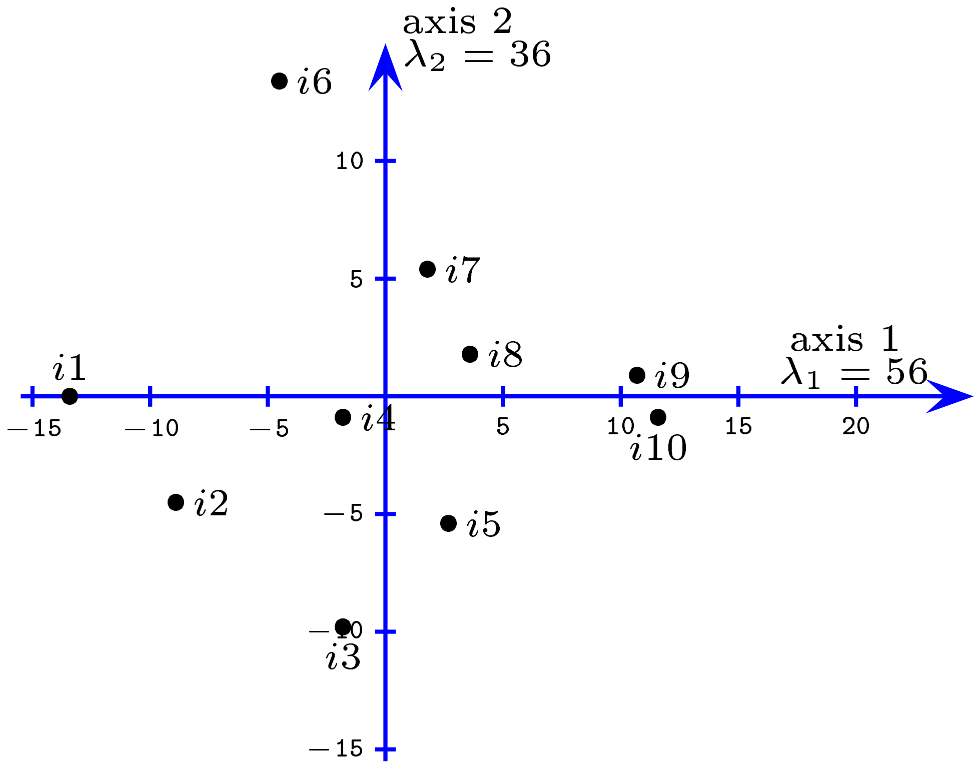
	$\mathcal{D}_1$	$\mathcal{D}_2$	$\mathcal{D}_3$	$\mathcal{D}_4$	$\mathcal{D}_5$	$\mathcal{D}_6$	$\mathcal{D}_1$
Variance	52	42.1	36.1	40.0	49.9	55.9	52

The directed line such as the variance of the projected cloud is maximum is called *first principal axis principal*.

Projected cloud = *first principal cloud*, its variance is called *variance of the axis* (denoted λ).
The firts principal cloud is the best fitting of the initial cloud by an unidimensionnal cloud in the sense of orthogonal least quares
Here, $\alpha = 63°$, $\lambda_1 = 56$.







Principal representation of the cloud (plane 1-2).

- *Quality of fit of an axis* or variance rate $\frac{\lambda}{V_{\text{nuage}}}$
- Principal coordinates of point define *principal variables*

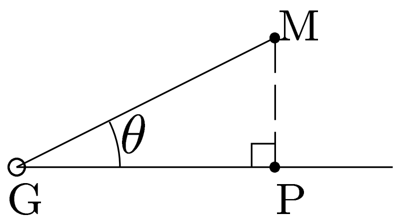
  with mean = 0 and variance = λ (eigenvalue)

  The principal variables are uncorrelated

- *Contribution of point i to an axis* : $\text{Ctr} = \dfrac{p\,(y)^2}{\lambda}$
- *Quality of representation of a point onto an axis* :

$$\cos^2\theta = \frac{\text{GP}^2}{\text{GM}^2}$$

Example: for $i2$, $\cos^2\theta = \dfrac{(-8.94)^2}{100} = 0.80$



- *Reconstitution of distances* :

$$d^2(i1, i2) = (-13.4 + 8.9)^2 + (0 - 4.47)^2 = 4.23 = (6.3)^2$$

Results of the analysis

$\lambda_1 = 56$ (variance of axis 1, eigenvalue).

$$\text{Variance rate : } \frac{\lambda_1}{V_{\text{nuage}}} = \frac{56}{92} = 61\%$$

Results for axis 1 ($\lambda_1 = 56$)

	Coordinates	Ctr (%)	squared cosines
$i1$	−13.41	32.1	1.00
$i2$	−8.94	14.3	0.80
$i3$	−1.79	0.6	0.03
$i4$	−1.79	1.3	0.80
$i5$	+2.68	3.6	0.20
$i6$	−4.47	3.6	0.10
$i7$	+1.79	0.6	0.10
$i8$	+3.58	2.3	0.80
$i9$	+10.73	20.6	0.99
$i10$	+11.63	24.1	0.99

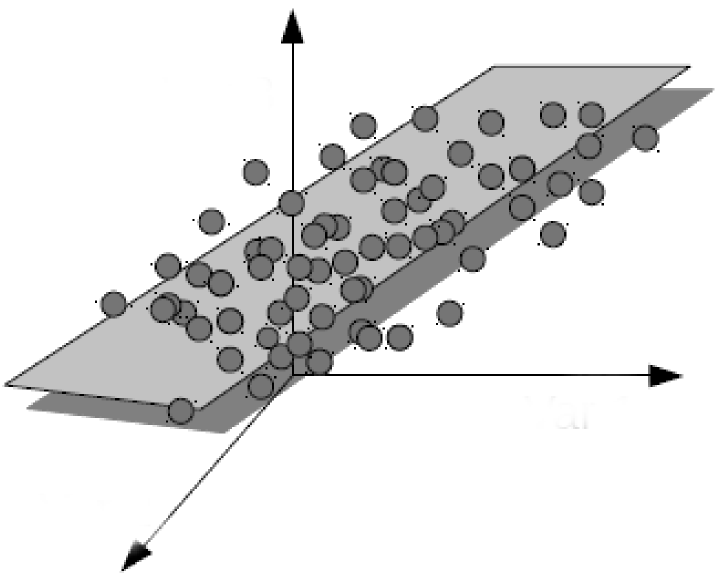
Results for axis 2 ($\lambda_2 = 36$)

	Coordinates	Ctr (%)	squared cosines
$i1$	0.00	0	0.00
$i2$	+4.47	5.6	0.20
$i3$	+9.84	26.9	0.97
$i4$	+0.89	0.2	0.20
$i5$	+5.37	8	0.80
$i6$	−13.42	50.0	0.90
$i7$	−5.37	8	0.90
$i8$	−1.79	0.9	0.20
$i9$	−0.89	0.2	0.01
$i10$	+0.89	0.2	0.01

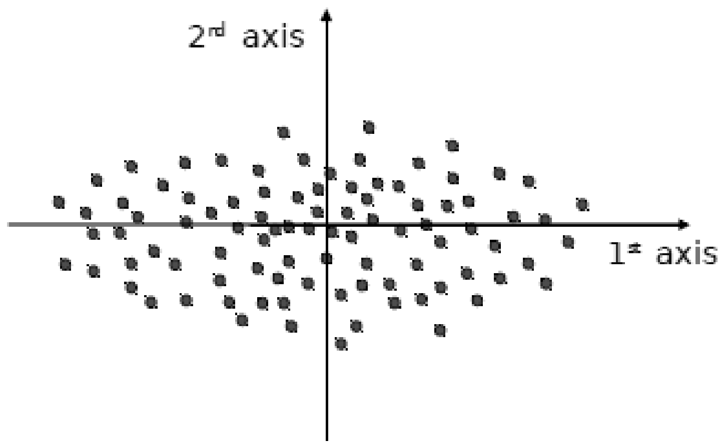
II.4 From a plane cloud to a Higher Dimensional Cloud

Heredity property: the plane that best fits the cloud is the one determined by the first two axes.

High dimensional cloud.



Low dimensional projection.



Properties

- $\sum_{\ell=1}^{L} \lambda_\ell = V_{\text{nuage}}$ (L denotes the dimensionality of the cloud).
- The principal axes are pairwise orthogonal.
- Each axis can be directed arbitrarily.
- The principal variables corresponding to distinct eigenvalues are uncorrelated.