Speaking of Geometry

A study of geometry textbooks and literature on geometry instruction for elementary and lower secondary levels in Sweden, 1905-1962, with a special focus on professional debates

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Abstract

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This dissertation deals with geometry instruction in Sweden in the period 1905-1962. The purpose is to investigate textbooks and other literature used by teachers in elementary schools (ES) and lower secondary schools (LSS) – Folkskolan and Realskolan – connection to geometry instruction. Special attention is given to debates about why a course should be taught and how the content should be communicated.

In the period 1905-1962, the Swedish school system changed greatly. Moreover, in this period mathematics instruction was reformed in several countries and geometry was a major issue; especially, classical geometry based on the axiomatic method. However, we do not really know how mathematics instruction changed in Sweden. Moreover, in the very few works where the history of mathematics instruction in Sweden is mentioned, the time before 1950 is often described in terms of "traditional", "static" and "isolation".

In this dissertation, I show that geometry instruction in Sweden did change in the period 1905-1962: geometry instruction in LSS was debated; the axiomatic method and spatial intuition were major issues. Textbooks for LSS not following Euclid were produced also, but the axiomatic method was kept. By 1930, these alternative textbooks were the most popular.

Also the textbooks in ES changed. In the debate about geometry instruction in ES, visualizability was a central concept.

Nonetheless, some features did not change. Throughout the period, the rationale for keeping axiomatic geometry in LSS was to provide training in reasoning. An important aspect of the debate on geometry instruction in LSS is that the axiomatic method was the dominating issue; other issues, e.g. heuristics, were not discussed. I argue that a discussion on heuristics would have been relevant considering the final exams in the LSS; in order to succeed, it was more important to be a skilled problem solver than a master of proof.

Keywords: geometry instruction, mathematics education, Sweden, textbook, curriculum, professional, elementary school, secondary school, history of mathematics education, history of mathematics, history of education

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Contents

Part A – Writing a history of geometry instruction

Introduction

An important event in the history of Western societies is the introduction of Hindu-Arabic numerals, the decimal number system, symbolic algebra and analytical geometry – processes that took place during the 15th, 16th and 17th centuries. These mathematical innovations paved the way for calculations and applications in science, trade, finance and navigation that were hard to execute only by means of classical Euclidean geometry and the Roman number system based solely on positive whole numbers.[1] Without any particular references, I think it is fair to say that these innovations have become indispensable parts of economics, engineering, science and other related practices. Moreover, this conflation has constituted a source for arguments regarding the content of school mathematics; courses in mathematics are, in some way, supposed to match applications in economics, engineering and science, but also everyday situations where you need arithmetic, algebra and coordinate geometry.[2]

In this perspective, courses in classical Euclidean geometry conveyed in an axiomatic style, e.g. via Euclid's *Elements*, appear to be a part of another context. If we consider common Swedish textbooks in elementary geometry from the 18th century to the early 1960's, the main point is the logical edifice

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[1] See for instance Swetz (1987), Dear (2001), Crosby (1997). Swetz (1987) discusses the importance of the introduction of arithmetic with Hindu-Arabic numerals in connection with the emerging merchant houses in Renaissance Italy. He underscores the merchants' growing interest in measurements of time, distance and capacity, but also their attempts to solve problems by means of numerical data, computations and empirical investigations.[Swetz (1987), pp. 291-295] Dear (2001) discusses the Scientific Revolution during the 17th century. He points out that the works of such pivotal figures such as Descartes and Newton not only caused an intellectual shift regarding the understanding of the universe; the mathematical methods in their dissertations also constituted a shift in how scientist worked in natural philosophy.[Dear (2001), pp. 168-170] Crosby (1997) adopts an even broader perspective and discusses how a new outlook on time, space and the physical reality developed in the countries of western Europe during the period 1300-1600. According to Crosby (1997), this new outlook was characterized by a predilection for visualizations and quantitative descriptions of the physical reality. Crosby (1997) discusses examples from art, music, bookkeeping, trade, navigation, cartography and astronomy. [Crosby (1997), pp. 185-195]

[2] See the background chapter of this dissertation for a description of the argumentation of the late 19th century and the first half of the 20th century.

of definitions, axioms, constructions, theorems and proofs. Furthermore, the ambition of several of the authors seems to have been to provide editions of Euclid's *Elements*. In these textbooks we find no, or very few, examples of applications. Calculations of lengths, areas and volumes are treated briefly or not at all.[3] By putting the logical structure at the fore, the textbook authors seem to have taken aim at purely logical aspects of mathematics. Indeed, a common argument for having courses in axiomatic geometry has been that they provide excellent training in reasoning.[4]

During the 19th and 20th centuries, courses in axiomatic geometry were criticized in several western countries. The critics downgraded the effects on the students' ability to reason and considered the courses too theoretical and inappropriate as preparation for education in technology and science.[5] However, extensive courses in axiomatic geometry, in some form, remained a part of the curricula in several European countries during the first half of the 20th century. In Sweden, extensive courses in axiomatic geometry were kept at lower secondary level until the early 1960's.

At least in the Swedish context, this may seem a bit awkward, especially if we consider the first half of the 20th century and the new curriculum of 1905. In this curriculum, sciences and mathematics together with modern languages were the major subjects at lower secondary level. Classical languages, on the other hand, were dropped completely. Moreover, lower secondary education contained just one program; this program should prepare the students for vocational educations and working life as well as further studies at upper secondary level and university. Thus, the course program in mathematics, axiomatic geometry included, functioned as a preparation for further studies at upper secondary level as well as various vocational educations. Specialized practical programs at the lower secondary schools were introduced in the 1930's; however, until the late 1940's, a great majority of the students, 90 percent or more, followed the curriculum in mathematics that included axiomatic geometry.[6]

The profile of the lower secondary schools was underscored by their name – Realskola – a name that was introduced in 1905. Here we may dis-

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[3] See for instance Strömer (1744), Wolff (1793), Lindman (1867), Asperén (1896), Vinell (1898), Sjöstedt (1936) and Olson (1940). The last edition of Strömer's geometry textbook was printed in 1884.

[4] See the background chapter in this dissertation.

[5] Of course, this type of criticism was conveyed well before the 19th century. And of course, there were geometry textbooks focused on applications before the 19th century as well. Kokomoor (1928a) and Kokomoor (1928b) provide am illuminative treatment of geometry and geometry instruction of the 17th century. A good example of a pre 19th century textbook in applied geometry is Clairaut (1744). This particular textbook was also translated into Swedish. For further details about arguments about mathematics instruction during the 19th and 20th centuries, see the background chapter in this dissertation.

[6] See the background chapter for a description of the Swedish schools system. The figure 90 percent is based on the statistics regarding the final exams of Realskolan that I display in Part E of this thesis.

cern an ambition among the politicians responsible to distance the new curriculum from past times; times when the humanities, especially Latin and Greek, dominated the curricula at secondary level, especially the course program leading to the prestigious state universities. However, this ambition encompassed not only politicians and leading actors in the public debate. Secondary school teachers in science, who took part in the public debate, can also be considered a progressive group in this respect. They not only wanted to reform the school system with respect to science; they also longed for a new outlook on society, science, and religion.[7]

In this perspective, the fact that extensive courses in axiomatic geometry were part of lower secondary education during the period 1905-1962 might appear to be a relic of the past. It can also be seen as an evidence of rigidity when it comes to school mathematics. In the school regulations, in the curricula and in the public debates the goals had changed, but the content of the courses remained as it was. Indeed, mathematics instruction in Sweden during the first half of the 20th century is often described in terms like "traditional," "static," and "isolated."[8]

The description of Swedish mathematics instruction as being traditional, static and isolated also stands out in comparison to how mathematics instruction in other Western countries is described. During the first decades of the 20th century, mathematics instruction was discussed and reformed in countries such as Germany and England. Several of the issues then discussed were indeed related to geometry instruction. For instance: What was the value of the axiomatic method in geometry instruction? What was its value in relation to education in general? Was it possible to adjust geometry instruction to the students' spatial intuition? Could geometry instruction develop spatial intuition? How could experimental teaching methods be integrated in geometry instruction? Moreover, the international exchange of ideas in the field of mathematics instruction increased when international journals and conferences devoted to mathematics instruction were organized for the first time at the beginning of the 20th century.[9] Thus, in comparison to what we know about mathematics instruction in other western countries, the impression of rigidity in Swedish school mathematics is amplified.

However, mathematics instruction in Sweden during the first half of the 20th century has not yet been investigated systematically. In this dissertation, I take a fresh look at elementary[10] geometry instruction in Sweden during the

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[7] For further details, see the chapter regarding literature on the history of Swedish mathematics instruction or the background chapters.

[8] See for instance Magne (1986), Håstad (1978) and Unenge (1999). Their standpoints are described in the chapter about literature on the history of Swedish mathematics instruction.

[9] These reforms are described in the background chapter below.

[10] By elementary, I mean the lower secondary schools as well as the elementary schools, i.e. Realskolan and Folkskolan.

period 1905-1962, a school subject and a period that in fact may not have been as traditional, static and isolated as previously believed.

Research plan

Purpose and questions

The purpose of this dissertation is to investigate textbooks and other literature that was used by teachers in elementary and lower secondary schools in connection to geometry instruction. In particular, I investigate and describe *professional debates* about elementary geometry instruction. During my investigations, I have set out to answer two main questions:

1) What arguments regarding *content*, *goals*, and *methodologies* occurred in the more comprehensive essays and articles on geometry instruction?

2) What was the *significance* of these arguments? This question is divided into two questions.

a) What was the *meaning* of the concepts used in the argumentation on geometry instruction? Or more specifically, what objects or phenomena did the persons mean by the crucial words and expression used in the argumentation.

b) What was the *relevance* of the arguments regarding geometry instruction? More specifically, to what extent did arguments on geometry instruction concern the practice of professionals, i.e. teachers, but also textbook authors and test constructors?

Scope of the dissertation – subject and time period

My motive for restricting my study to geometry instruction and the period 1905-1962 is that this particular school subject was discussed in several Western countries during the period. Moreover, geometry filled a great part of the curricula in mathematics in Sweden and other Western countries during this period.[11]

My motive for restricting my study to the period 1905-1962 is linked also to the changes of Swedish primary and secondary education that took place during the period. Apart from the establishment of Realskolan in 1905 and the growing importance of Realistic education at secondary level, four other sequences of changes stand out in particular:[12]

- the expansion of the number of students at secondary level
- the integration of primary and secondary education
- women's access to higher education

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[11] See the background chapter for further details.
[12] These aspects of the Swedish school system are described in the background chapter below.

- the secularization[13] of the school system

All these changes concern the selection of students and the content of the courses, i.e. basic features of a school system.

The reason to put 1962 as an endpoint is that axiomatic geometry disappeared from the Swedish course plans in connection with the introduction of Grundskolan in 1962. Grundskolan is the general compulsory school that replaced the previous school types, i.e. the elementary schools, the lower secondary schools and the girl schools.[14] The choice to put this year as the endpoint for my study is also linked to the major reformation on school mathematics that took place in Sweden and other countries during the 1960's.[15]

Sources and selection – the professional literature

In many ways, school regulations, course plans, textbooks, teacher journals, teaching literature, and official school reports constitute the natural sources for a study on the history of mathematics instruction. Perhaps the most basic reason for this is that these texts are the remains of mathematics instruction of past times. However, to choose from this set of sources and to investigate them is not as straightforward. I would say that these sources represent a rather chaotic collection of voices about what school mathematics was supposed to be; authors with different backgrounds and aims, such as school officials, mathematicians, researchers in education, textbook authors, ordinary teachers (not so often though), politicians, industrialists, intellectuals, etc, express their ideas about how things ought to be arranged. Here, I also include the textbooks as the voice of its author. As I see it, a textbook primarily expresses the standpoints of its author and should, besides that, be considered a recommendation regarding what the teachers should teach; textbooks do not reveal what parts the teachers chose to teach and how they taught. Textbooks can be quite deceptive in that way.

Having this situation in mind, I think it is important to decide who have been the readers and the users of the sources that you investigate. In order to distinguish these readers and users, I have considered the school system as a set of different groups of expertise. Examples of such groups are school politicians at the national or municipal level, officials with the central school authorities, researchers in education, or teachers in different subjects and school forms. Primarily, I separate these groups by their working tasks and their methods to solve these tasks; I think of these tasks and methods as the basic components that join the members of groups of expertise.

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[13] I am here referring to the decreasing administrative responsibilities of the church as well as the changing contents and goals of the courses in Christianity, later on religion.
[14] See the background chapter for a brief description of the Swedish school system of that time.
[15] See the background chapter for further details.

On the basis of this distinguishing of groups of expertise, I am interested in texts that experts have used as they practice their expert skills, especially the texts that describe the main tasks and methods.

The group that I focus on in this thesis comprises persons whose profession was to communicate mathematics in the elementary schools and the lower secondary schools – they are the experts, the readers, and the users I am interested in. By professionals in this case, I intend not only teachers, but also textbook authors and test constructors. Consequently, the main sources of this thesis are texts that have been written for and used by professionals in connection with elementary geometry instruction. I denote these texts the *professional literature* on elementary geometry instruction. In practice, my main sources are course plans, textbooks, final exams, and articles and essays with a special interest in mathematics instruction.[16]

This group of professionals and the texts they read and used in connection to their profession are relevant if we want to understand the production and reproduction of school mathematics, but also in what respect these processes have changed. In the end, the schools are the essential part of the educational system where education is going on, and it is the work of teachers, textbook authors and test constructors that upholds this practice.[17]

The professional debate

One part of my investigations have been to describe a professional debate on elementary geometry instruction; a debate where the main problems and methods of a professional practice were defined and motivated. Of course, we cannot consider a debate like this an imprint of teachers' thoughts and views in general. On the other hand, this does not entail that an investigation of such a debate reveals nothing more than the views of the debaters. My position in this matter is to regard the professional debate as a source of potential arguments and an incentive for actions, arguments that may be used by the common teacher to convince her fellow colleagues about some issue. By this, I do not mean that one had to comply with certain directives or arguments. You could, of course, also criticize, reject, or disregard the whole set of directives and arguments, or parts of it. But, even though you ignore the debate, other professionals may have put you in relation to it.

The main items of my investigations in this respect have been a number of articles and essays in journals and treatises specialized in mathematics

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[16] The sources are accounted for in chapter N.

[17] The method to focus on groups that on a professional basis spread or use scientific knowledge is described by for instance Widmalm (1999). The research object that he then refers to is the interplay between technology, industry, politics, and science during the 20th century. [Widmalm (1999), p. 13] He also accentuates the need to investigate not only the discovery of scientific knowledge, but also how it was applied and adapted to the needs of everyday life. [Widmalm (1999), pp. 18-20]

instruction.[18] These texts were first and foremost intended for the working teachers and concerned their professional practice either in the elementary schools or in the lower secondary schools. Thus, it was not a public debate, but a debate where professionals turn to other professionals of the same practice, although more general issues on education and society were mentioned.

In comparison to other texts where mathematics instruction was treated, for instance curricula, textbook reviews, or contributions to public debates on education, these articles and essays are more exhaustive when it comes to geometry instruction. By 'more exhaustive', I mean that crucial concepts and expressions, as for instance *visualizability* or *training in reasoning*, were not used only as catchwords, but explained more fully.

The authors of these texts occupied central positions in the group of professionals engaged in mathematics instruction. Among these authors we find two teacher educators, two leading school officials, one constructor of curricula, one constructor of finals exams, several editors of teachers' journals, several textbooks authors, and authors of books on teaching methods used in teacher training. Some of them filled two, three and even four of these roles. Apart from that, all of them worked or had worked as mathematics teachers. However, we should not consider them as regular teachers. They rather constituted an elite group who set the agenda for the professional debate on mathematics instruction.

During the investigation of the articles and essays, I have focused on three types of directives and arguments regarding elementary geometry instruction, which I denote *content*, *goal*, and *methodology*. These categories correspond to the basic didactical questions: what? why? and how?

By *content*, I mean directives and arguments concerning the elements that should be included in the geometry courses, e.g. mathematical concepts, theorems, formulas, other propositions, algorithms, methods, applications, etc. In this perspective, a textbook constitute a directive, or a recommendation, regarding what the teachers are supposed to teach.

The *goals* are directives or arguments regarding what geometry instruction is supposed to achieve. By goals, I do not only intend that the students are supposed to master geometry, but also other types of goals, for instance:[19]

- Moral goals – Students are usually subjects of some kind of moral schooling, for instance good manners, piety, patriotism, equality between the sexes, environmental awareness, economic awareness, democracy, critical thinking, etc.
- Epistemological goals – These goals are related to conceptions of what it means to know and learn something. One example is the

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[18] I describe these articles and essays in the beginning of Part C.
[19] These categories have occurred to me as I have worked through the source material, and the names are my own. In this dissertation, they serve as an example of different types of goals.

current Swedish syllabus, where the concepts *facts*, *skills*, *understanding* and *belief* are used in discussions on knowledge and what the students are supposed to learn.

- Functional goals – These goals are related to conceptions of how the students are supposed to use their knowledge, for instance in further education or to solve tasks in every day life. In this category, I also include directives or arguments regarding for which groups of students the education is intended.

A *methodology* is a set of arguments regarding how and why the geometry courses are supposed to be communicated to the students in order to achieve the goals. Hence, by a methodology, I do not merely intend teaching methods to be applied in the classroom. This category include arguments regarding why and how and concepts shall be explained and defined; how theorems should be introduced, explained or proved; how exercises should be designed; how tests should be designed; how textbooks should be designed; which symbols ought to be used, etc.

However, there are some aspects of the source material and the concept of professional debate that need to be clarified. If we understand a professional debate as an incentive for actions and a source of potential arguments, I think it is relevant to ask about the significance of the arguments. What did the arguments mean to the professionals, and what was their relevance?

The significance of a professional debate

A critical aspect of the sources investigated is that they mainly constitute *ideals* regarding mathematics instruction; they contain directives and arguments about how things *ought* to be arranged. At this point, I am referring not only the articles and essays of the elite, but also curricula, textbooks, other articles and tests in general. Thus, when making an investigation of professional literature on mathematics instruction you run the risk of describing what the professionals *may* have talked about or which textbooks they *may* have used.

My point is that these sources do not reveal the *significance* of the directives and arguments. Here, I distinguish between *meaning* and *relevance*.

In curricula, textbooks, articles, teaching literature, etc., the authors did not very often elaborate upon how they, or others, understood concepts and propositions, i.e. they did not reveal a more precise *meaning* of the concepts and propositions. This is by no means surprising since that was not their prime motivation for writing this type of text, and the discussions were kept at a general level. This has been my reason to focus the investigations on more exhaustive treatises. But neither have the authors of these texts explicated their full understanding of concepts and arguments.

The other aspect is that the source material does not provide explicit information about the *relevance* of directives and arguments; i.e. in what re-

spect did certain directives and arguments concern the practice of teachers, textbook authors, and test constructors. The directives and arguments were perhaps more of a facade.

Niss (2001) discusses a similar issue in an article about the goals of mathematics instruction.

> Even if official or semi-official goals are in fact established in explicit terms in accessible documents, it is far from certain that these goals are the *real* ones, those that underpin mathematics education and guide teaching and learning. Oftentimes goals are formulated 'post festum' in order to embellish the curriculum or the syllabus expositions, or to provide a persuasive preamble to politicians, administrators, employers, parents, colleagues in other subjects, or to be a *memorandum* to the teachers who are to implement the curriculum. Moreover, it frequently happens in periods of curriculum reform that the explicit goals are changed while the curriculum remains largely unchanged. Or the converse, for that matter, that the curriculum is changed while the official goals remain the same.[20]

As an illustration of this problem, we can consider a textbook in mathematics intended for the elementary schools, i.e. Folkskolan, printed in 1931. In the foreword, the authors declare quite briefly that explanations and exercises are based on *åskådlighet*, a noun often translated as *spatial intuition*. However, another translation is *visualizability*, a translation based on the verb *skåda*, which means *to see* or *to watch*.

In the current curriculum, it was also established that mathematics instruction should follow a principle regarding *åskådlighet*.

> 1. *Visualizability* [~åskådlighet] should as far as possible be aimed at during [mathematics] teaching. Measuring and weighing should for instance be considered the foundation for the calculation of measures and weights, and the operations of calculation should, when possible, be made visual [åskådliggöras] by counting objects.[21]

This recommendation reveals a bit more. It indicates that *åskådlighet* means *to make something visual* by referring to the students' experiences (measuring and weighing) or some real objects. From this, we might infer that illustrations were used frequently in the textbooks.

However, in comparison to any modern textbook intended for the grades 4 to 6, the illustrations are quite few.[22] Take for instance the introduction of percentage in grade six; this was done without any illustrations what so

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[20] Niss (1996), pp. 17-18. For a Swedish version of the article see Niss (2001). The quote is located on pp. 58-59 in Niss (2001).
[21] Kungl. Skolöverstyrelsen (1919), p. 67: "1. Vid undervisningen bör så långt som möjligt *åskådlighet* eftersträvas. Så t. ex. böra mätningar och vägningar läggas till grund för räkningen med mått- och viktsorter, och räkneoperationerna böra, då så lämpligen kan ske, åskådliggöras genom räkning med föremål." [The italics are in the original curriculum.]
[22] Asperén et al (1931), p. 3

ever.[23] In the chapters on geometry, the number of illustrations was greater. Yet, the illustrations were mainly used as parts of ostensive definitions of concepts – this is a rectangle – or when constructions were described, e.g. the bisection of angels or the generation of perpendiculars.[24] My point is that the authors could not have used fewer illustrations unless they had skipped all types of illustrations. On only a few occasions were the illustrations used to explain the meaning of a proposition in a different manner. For example, in connection with the proposition, *a straight line is the shortest distance between two points*, the students were supposed to measure the lengths of a straight line and a crooked line that shared the same endpoints; these lines were also depicted in the book. The purpose of the picture and the measuring task seems to have been to provide a deeper understanding of the proposition.[25]

Thus, the significance of the formulations about *åskådlighet* is not obvious. Did they have any relevance? Was it just lip service paid by the authors or did they take this idea seriously? Nor is it obvious what the concept *åskådlighet* meant to the authors. Perhaps *åskådlighet* was important to them, but it did not mean that textbooks had to be crammed with illustrations.

Nevertheless, I do find it possible to investigate the *significance* of directives and arguments, at least to some extent. An important part of my approach to this problem has been to consider the mathematics in textbooks and final exams in the light of directives and arguments about geometry instruction. At first glance, the textbooks appear very much the same; definitions, theorems, formulas, proofs, explanations, exercises and illustrations are quite similar, if not identical. Each year, the exercises on the final exams look pretty much the same. Still, upon closer inspection you notice variations. If we compare these variations with the arguments in curricula, articles, and methodological literature, it is clear that these variations are not accidental.

Thus, by comparing directives and arguments about content, goals and methodology with textbooks and final exams, we receive a good picture of how textbook authors and test constructors acted relation to these directives and arguments. Moreover, we get a picture of how the textbook authors and the test constructors understood these directives and arguments. Since curricula, journals, teaching literature, textbooks, and final exams were intended for persons that took part in the same professional discussions; this comparison gives an indication of how teachers conceived of geometry instruction. More specifically, we get a picture of what the teachers talked about, albeit not their standpoints in these issues.

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[23] Asperén et al (1931), pp. 68-76
[24] Asperén et al (1931), pp. 50-68.
[25] Asperén et al (1931), pp. 50.

In order to achieve a better estimation of the relevance of directives and arguments among the teachers, I have gathered statistics on the textbooks that were available. The choice of textbooks also gives an indication about the teachers' reception of the arguments about geometry instruction. For example, the choice between textbooks that followed Euclid's *Elements* very closely and textbooks that deviated from the *Elements* quite distinctly provides a rough approximation of the teachers' standpoints in the discussions on textbooks.

A source that exposes what the teachers were doing is the national final exams; they provide a good indication of what kind of exercises the teachers and the students grappled with, since we know that the students took the tests and that the teachers corrected them. Most likely, such exercises were treated by a majority of the teachers during lessons as well. Other types of exercises may of course have been treated, but I have not made any attempts to investigate the occurrence of such exercises. Principally, because it is too difficult to determine which other types of exercises were used. The textbooks did contain only definitions, axioms, and theorems, but there were booklets containing supplemental exercises. On the other hand, it is difficult to determine which exercises the teachers chose to work with during lessons and how they used them.

In my investigations, the national final examinations constitute a benchmark for the contemporary directives and arguments regarding the goals of geometry instruction. For example, training in reasoning was a central issue in several articles, but the discussions were kept on a general level. Instead, the final exams give an idea of what this training included. The results at the national final exams also give an indication as to whether or not the teachers succeeded in their teaching and in attaining the goals. An interesting source in this respect is the annual evaluation reports about the final exams where also the teachers' corrections of the final exams were commented on and evaluated. Another source related to the final exams is the actual student papers. These sources related to the final exam have been investigated as well.

Regarding the investigation of arguments and directives, my point of departure has been to describe the professionals' *explicit* statements on geometry instruction. The statements have been taken at face value, and I have not been trying to reveal some hidden agenda or underlying policy during my search for arguments and directives. The significance of the arguments and directives has been investigated in textbooks and final exams.

The value of an approach where you consider the significance of certain concepts, arguments, or theories and not just concepts, arguments, and theories per se has been emphasized by, for instance, Widmalm (1999);[26] he refers to studies on the history of learning and education in Sweden. Schubring

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[26] Widmalm (1999), p. 11

(1989) underscores the value of this kind of approach in investigations of the history of mathematics and the history of mathematics instruction.[27] His work on the mathematics teacher profession in Prussia during the period 1810-1870, Schubring (1991), is very much in line with this intention.

Outline of the dissertation

Besides this part of the dissertation (Part A) the dissertation is divided into six more parts: B, C, D, E, F and G

Part B contains two background chapters. In the first chapter, I describe some basic features of the Swedish school system during the period 1905-1962. In the second chapter, I describe the history of geometry instruction in Western countries, mainly in Prussia/Germany and England.

In Part C, I address a number of exhaustive articles and essays on elementary geometry instruction. In this part, I also describe the passages of the curricula that concerned geometry instruction. This description also includes the time plans. The main purpose of this part is to answer the first question: What directives and arguments regarding content, goals, and methodologies occurred in the more exhaustive essays and articles on geometry instruction during the period 1905-1962? Part C ends with a chapter where I summarize the main directives and arguments about elementary geometry instruction. Here, I also foreshadow some answers to the second question about the significance of the arguments.

In Part D and E, I treat the most common textbooks used at the common and lower secondary schools, i.e. Folkskolan and Realskolan, during the period 1905-1962. In Part F, I treat the final exams in mathematics for Realskolan together with correction forms and the reports on how the students as well as the teachers had performed on the tests. The main purpose of Parts D, E and F is to answer the second question: What was the *significance* of the arguments on geometry instruction?

Part G constitutes an epilogue. Here, I summarize my main results and relate them to previous literature where mathematics instruction in Sweden and other countries is treated.

Literature on the history of Swedish mathematics instruction

To my knowledge, there are very few treatises on the history of Swedish mathematics instruction. If we consider more extensive treatises that are

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[27] Schburing (1989), p. 171

based on primary sources[28] and that were authored during the last 100 years, I can think of only one: Hatami (2007). In this thesis, The Rule of Three in Swedish textbooks during the period 1600-1960 is investigated. To be more exact, he considers The Rule of Three as a manifestation of rhetorical mathematics;[29] in this perspective he discusses how The Rule of Three has been introduced and explained in textbooks. I return to this treatise below.

However, despite the small number of treatises, there is indeed an interest in the subject. If we consider Swedish didactic or pedagogical treatises on mathematics education, brief historical backgrounds are sometimes included in these works.[30]

A common feature of these backgrounds is that they pay considerable attention to the great school reforms of the 1960's, i.e. the introduction of Grundskolan and the New Gymnasium (the new school type for upper secondary school). In these backgrounds, the authors point out that these reforms were accompanied by radical changes in mathematics instruction, e.g. the abandonment of classical Euclidean geometry, new teaching methods, and later on the introduction of the New Math. They also point out that these reforms were accompanied by major programs for in service education of mathematics teachers.

Another common feature of these historical backgrounds is that they contain explanations of why mathematics instruction was reformed during the 1950's and the 60's. Three types of reformist arguments regarding mathematics are put forward.

- School mathematics was considered old-fashioned from a scientific point of view. Therefore, the courses were changed.[31]
- School mathematics was considered old fashioned in relation to the needs of a modern society. These needs could be in science or technology, but also so-called everyday situations. Therefore, in order to provide knowledge more suitable for such areas, the courses in mathematics were changed.[32]

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[28] By primary sources in this case I intend texts that were used in connection to mathematics instruction, e.g. course plans, textbooks and teaching literature.

[29] Hatami (2007), p. 8. Retorical mathematics is defined in the following way by Hatami (2007): "In rhetorical mathematics, mathematical problems are solved by means of ordinary language. By stepwise reasoning, instead of a ready made algorithm or a mathematical model, the solution is reached. A mathematical model can be an equation that is solved by means of certain rules." [Hatami (2007), p. 14]]

[30] See for instance Håstad (1978), Kristiansson (1979), Hellström (1985), Magne (1986), Hedrén (1990), Löthman (1992), Hellquist (2000), Bjerneby Häll (2002), Samuelsson (2003) and Nilsson (2005)

[31] Hellström (1985), p. 12; Hellquist (2000), p. 225, Kristiansson (1979), p. 2; Nilsson (2005), pp. 29-34

[32] Kristiansson (1979), p. 1; Nilsson (2005), pp. 37, 41-43, Samuelsson (2003), pp. 40-41; Håstad (1978), pp. 114, 117; Hedrén (1990), p. 23; Löthman (1992), p. 39

- Pedagogical and psychological research had brought a new awareness about mathematics instruction and learning, which resulted in a reformation of the courses and new teaching methods.[33]

By putting forward these arguments as a source of change, the authors of the historical backgrounds links the changes in school mathematics to changes in society and science. This relation is underscored with even greater emphasis by the Swedish pedagogue Sellander (2001) in an essay on Swedish school mathematics of the 18th century:

> For centuries school mathematics has evolved from a formal education and a deductive method to today's functional orientation with elements of an inductive method. Early on, one considered mathematics, along with Latin, important for the students' ability to think and make judgments, and mathematics provided useful training in systematic working procedures and clarity of thought. … During the 18th century, a long mathematical tradition based on Euclid's geometry was complemented by arithmetic; thereby, mathematics was more adapted to the new needs of the era that followed upon the expansion of shipping and trade. … During the late 19th century, with the breakthrough of industrialism and the formation of modern schools, with industrial chemistry and new sources of energy, with railroads and national time standards, with the organization of labor and capital and the formation of the modern national state, school mathematics was renewed once again, as the practical relevance of school mathematics was accentuated even more.[34]

However, the historical backgrounds and Sellander's essay leaves a couple questions. Why did the great changes of mathematics instruction in Sweden take place only in the 1950's and 60's? Why did courses in axiomatic geometry, courses that contained few applications, remain a part of the curricula of Realskolan? I mean, the breakthrough of industrialism that Sellander describes took place in the late 19th century.

However, in comparison with the concern about the 1950's and 60's, the half century before 1950 is treated very briefly in the historical backgrounds, most times not at all. In cases where mathematics instruction of this period is given a slightly longer treatment, it is often described in terms of its tradi-

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[33] Hellström (1985), pp. 11-12; Kristiansson (1979), pp. 3-4; Nilsson (2005), pp. 29, 32; Samuelsson (2003), pp. 35-36

[34] Sellander (2001), pp. 41-42: "Matematiken i skolan har under århundraden utvecklats från ett formalbildande och deduktivt arbetssätt till dagens funktionella inriktning med inslag av induktiva arbetssätt. Tidigt ansåg man att matematik, vid sidan om latinet, hade betydelse för elevernas tankeförmåga och omdöme, och att matematik var nyttig träning för att man skulle lära sig ett systematiskt arbetssätt och tankeklarhet. ... En lång matematisk tradition baserad på Euklides geometri kompletterades under 1700-talet med aritmetiken, och matematiken blev därmed mer anpassad till den tidens nya behov som följde med den växande sjöfarten och handeln. ... Under 1800-talets slut, med industrialismens genombrott och den moderna skolans formande, med kemiindustri och nya energikällor, med tågtrafik och nationellt enhetliga tider, med organisering av arbete och kapital och formandet av den moderna nationalstaten skedde åter en förnyelse av skolmatematiken, då matematikens praktiska relevans betonades än mer." My translation.

tionalism, isolation, and stagnation. In a report on theories on common education and mathematics instruction in Sweden, Magne (1986) claims that mathematics instruction at the common level was completely unaffected by the reform movements in other Western countries before 1950.[35] However, he does suggest that mathematics instruction at the secondary level was influenced by international movements.[36] Nonetheless, in the same report, Magne points out that the international debates on geometry instruction around 1900 never reached Sweden, nor did the international debates on algebra instruction during the 1920's.[37]

In his doctoral thesis in mathematics education, Håstad (1978) takes it even further:

> If we must mention the force that has played the leading role in the development of mathematics instruction, the answer is simple: <u>tradition</u>. However, there are a number of other "persons in power" that have possessed important supporting roles. The study of their influence is crucial. And how important is tradition? In order to come to grip with its role [the role of tradition], I make the following simplification. <u>The mathematics instruction that took place up to the 1950's should be considered tradition</u>. The plausibility of such an assumption is compellingly vindicated by the fact that mathematics instruction has been relatively static during a long period of time and that only minor modifications have taken place during the previous decades.[38]

In an essay on school mathematics, Unenge (1999) summarizes his experiences of Swedish mathematics instruction before 1960 in a similar way.

> Well until the late 1950's, mathematics instruction was more or less unchanged. The way I was taught as a student in Realskolan was the way I taught my students in Realskolan 15 years later.[39]

Unenge began working as a teacher in 1952.

These descriptions regarding the 1950's and 60's are by no means wrong, as long as we are conscious about what aspects of the situation they describe. I have no doubts about the descriptions of the reformist arguments and the changes in the curricula; undeniably, these arguments were a part of the debate and the changes in the curriculum did take place. The problematic aspect is that only the arguments of the 'winning team' are mentioned – the arguments of those whose wishes came true in the reforms of the 1960's. The critics of the reforms and their counter arguments are not considered at all. In fact, from the historical backgrounds you cannot tell whether there were any critics. It is almost as if the changes that took place followed some

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[35] Magne (1986), p. 6
[36] Magne (1986), p. 6
[37] Magne (1986), p. 11
[38] Håstad (1978), p. 134. The underlining as well as quotation marks are original.
[39] Unenge (1999), pp. 24-25

kind of order related to the industrialization of Swedish society. Another problematic aspect is the brief descriptions of the decades before 1950 as being traditional and static. These depictions also appear to be the view of the 'winning team' and linked to the arguments about mathematics instruction being old-fashioned.

An interesting aspect of Magne's, Håstad's, and Unenge's descriptions of Swedish mathematics instruction prior to 1950 is their relation to research about the history of the Swedish school system. Their claims about traditionalism, isolation, and stagnation do not tally in any obvious way with a recent study on the debates regarding curricula in science and mathematics during the period 1905-1962. Lövheim (2006) characterizes the secondary teachers in science that took part in this debate as a progressive group, a group that longed not only for changes of the curricula, but also more general changes of education and society. He even describes them as the "undergrowth of the new enlightenment" working anonymously behind leading politicians and intellectuals.[40]

Did this group of progressive teachers really ignore the international debate on how to change mathematics instruction? Did none of these persons make any attempts to change mathematics instruction? Why did axiomatic geometry remain part of the mathematics courses in Realskolan? After all, mathematics was one of the major school subjects and science and technology contained a lot of mathematical applications. If the claims of Magne, Håstad, and Unenge are right, they provide an important perspective on Lövheim's results; in the public debate on education and society the teachers in science were progressive, but they were not willing to rethink their views on either the content of the courses in mathematics or the ways to communicate the content.

Actually, we do know about attempts to change mathematics instruction in Sweden during the 20's and 30's.[41] However, investigations of these attempts are quite brief, with one merely pointing out that some educators argued that mathematics instruction ought to change. What these attempts consisted in or in what respect they had any influence has not been investigated.

In this context, I want to return to the thesis of Hatami (2007). He shows that the treatment of The Rule of Three in the textbooks did change. Of interest for this thesis, different approaches to the subject were present during the 19th and 20th centuries.[42] This is a result that disagrees with the claims about traditionalism and stagnation before 1950.

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[40] Lövheim (2006), pp. 88-89.
[41] See for instance Johansson & Wistedt (1991a), Johansson & Wistedt (1991b), Kilpatrick & Johansson (1994), Nilsson (2005) and Magne (1986). However, Magne (1986) describes the impact of these attempts as limited.
[42] Hatami (2007), pp. 185-198.

Hatami (2007) also discusses the changes of school mathematics in the 1960's. He argues that the treatment of the The Rule of Three in a rhetorical manner included a dimension of elementary mathematics that is quite rare today. According to Hatami (2007), the rhetorical side of basic arithmetic with fractions has been neglected since the 1960's when The Rule of Three disappeared from course plans.[43]

As I see it, such observations about missing components in today's school mathematics are hard to make if you do not know what to look for. Hopefully, a thesis like the present one, addressing mathematics instruction of past times, will provide some perspective on present-day discussions on mathematics instruction. This might be at the research level, at the political level, in teachers' training, or among teachers. In this respect, the historical backgrounds discussed in this chapter are particularly interesting since they reflect how some researchers in mathematics education consider their research object. Considering the fact that only the reformist arguments of the 1950's and 60's have been documented, these historical backgrounds might give us reason to be a bit critical about how we normally conceive and pose research questions about mathematics instruction.

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[43] Hatami (2007), pp. 17-18, 199-203

Part B – Historical background

Introduction

This historical background contains two chapters.

In the first chapter, I put forward changes in the Swedish schools system during the period 1905-1962. In particular, I describe the position of Realistic education within the school system, the integration of primary and secondary education, girls' access to secondary education, and the increasing number of students at the secondary level. In order to get some perspective, the preceding century is treated as well.

In the second chapter, I describe influential movements and arguments regarding mathematics instruction in some Western countries during the period 1905-1962. Most emphasis is put on geometry instruction and the situation in Prussia/Germany and England. The reason for this restriction is the close cultural bonds between Sweden and the German-speaking countries during the 19th and early 20th centuries.[44] At the same time, English reformists did also provide arguments to the Swedish school debate during the late 19th century.[45] In this chapter as well, I include the preceding century in order to bring some perspective.

The backdrop for these changes in the school system is of course the transformation from an agricultural society to an industrial society and the demands for democracy, extended and equal civil rights, and religious, political, and economic freedom.

Throughout this dissertation, I use the notions Humanistic and Realistic education. These notions are taken from a work of the French sociologist Durkheim on the history of educational thought from the Middle Ages to the early 20th century. Here he identifies two fundamental theories regarding what a curriculum for secondary schools ought to comprise: he denotes them the educational theories of the Humanists and the Realists.[46] I understand the notions Humanistic and Realistic educations in the following way. The ultimate purpose of a typical Humanistic education is to convey knowledge about man and his reasoning, morals, religion, and culture. Such a curriculum is dominated by languages, rhetoric, logic, dialectic, literature, and art. The ultimate purpose of a typical Realistic education, on the other hand, is to

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[44] See for instance Richardson (1963), p. 131
[45] See for instance Richardson (1963), pp. 148-156
[46] Durkheim (1977), pp. 265-291.

convey knowledge about nature and things; such a curriculum is dominated by the sciences, but also engineering and economics.

Changes in the Swedish educational system

Humanistic and Realistic educations in Sweden

At the beginning of 19th century, the secondary schools in Sweden, i.e. Läroverken, comprised two course programs: one Humanistic and one Realistic program. The Humanistic program was conveyed at the Grammar schools (grades 3-8) and later on at the Gymnasiums (grades 9-12); the Realistic program was conveyed at the so-called Apologists schools (grades 3-8). The functions of these programs were quite different, as the Humanistic program should prepare the students for further studies at the state universities. The Realistic program, on the other hand, did not provide access to the state universities; it should prepare the students for more advanced vocational studies or more advanced tasks in working life.[47]

Throughout the 19th century, critics argued that Humanistic education was too dominant in the secondary schools. In particular, the value of the extensive courses in Latin and Greek was questioned; courses that were required for further studies at the state universities. One demand was that a Realistic program at Läroverken, with few lessons in classical languages or none at all, should provide entrance to the state universities. Moreover, some critics wanted a better and longer Realistic program with more advanced courses in mathematics and science along with modern languages. This should provide a better preparation for qualified vocational educations in engineering, agriculture, trade, navigation, pharmacy and military matters.[48] During the last decades of the 19th century, the critics were inspired by works by John Stuart Mill, Herbert Spencer, Thomas Huxley and Alexander Bain, a group of English philosophers that advocated Realistic school programs. This type of

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[47] Sjöstrand (1965), pp. 312-319: According to the school regulations of 1820, Läroverken comprised three programs: Apologists schools (year 3-8), Grammar schools (year 3-8) and Gymnasiums (year 9-12). The Apologists schools conveyed a general civic education that comprised the subjects Swedish, mathematics, German, French, history, law, politics, geography, natural history and Christianity. There were no courses in classical languages. Studies at the Apologists schools did not lead to higher education at university. The way up to university comprised studies at Grammar school and Gymnasium. Apart from a general civic education, these schools were to provide an education needed for the cultivation of scientific knowledge and state offices. The education at the Grammar schools did include similar subjects as the ones at the Apologists schools; however, the dominating courses were classical languages and mathematics. The first Latin course began in grade 3 and the first Greek course began in grade 4. The education at the Gymnasium was a bit more diversified and included more courses in modern languages, physics, natural history, botany, music and gymnastics.
[48] Sjöstrand (1965), pp. 221-240.

education was supposed to provide more useful practical knowledge for every day life as well as working life.[49]

A basic argument against extending Realistic education at secondary and university levels was linked to a conception where the state cared for the needs of the state; according to this view, the prime task of the secondary schools and the state universities was to educate officials for the state administration, the state universities, and the church. The Realistic programs, which were believed to benefit private and commercial interests in the first hand, should not be a concern of the state.[50] Historians also point at moral policies as a source for arguments against Realistic education; the advocates of a Humanistic education often mistrusted science and Realistic educations and put it in connection with a destructive materialism.[51]

The leading argument for keeping Latin as a major subject in secondary education was that it provided training in logic and a formal and general knowledge about languages.[52]

Gradually, the demands of the reformists gained influence. Still, the Realistic program did not provide eligibility to the universities until 1891. Before that, the basic academic degrees at the state universities, including degrees in science, required university courses in Latin. However, students in the Realistic program could enter technical universities and other advanced vocational educations in agriculture, trade, navigation, pharmacy, and military matters. During the 19th century, several institutes for vocational education in these areas were being founded;[53] institutes that later on turned into uni-

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[49] Richardson (1963), pp. 148-156

[50] See for instance Florin & Johansson (1993), pp. 105-120

[51] See for instance Richardson (1963), pp. 80-84, 132

[52] See for instance Richardson (1963), pp. 122-131

[53] Sjöstrand (1965), pp. 193-198, 214, 237, 321-333: In the school reform of 1832, the Realistic programs in some, but far from all, secondary schools were extended all the way up to the level of the Gymnasium. During the 1850's, the Apologists schools and the Grammar schools were replaced by lower Elementary schools (grades 3-8) and the Gymnasiums constituted the upper Elementary schools (grades 9-12). Moreover, both the lower and upper schools contained Humanistic and Realistic programs. During the first two grades of the lower schools, the students took the same courses. This meant that the first Latin course was postponed until grade 5. In 1869, grade 3 was dropped in the lower schools and in the 1870's it was regulated that all students should take the same courses in grades 4 to 6. Thus, the Latin courses at the Classical program began in grade 7, whereas the students at the Real program took courses in English. The only foreign language in grades 4 to 6 was German. The school reform of 1878 was however a small success for the reformists, since the Humanistic program was divided on the "classic" and the "half classic" program. Later on these programs were renamed the A and B programs. The A program retained great parts of the previous syllabus while Mathematics and English were added to the B program at the expense of Greek, which was dropped completely. Even though the Realistic program still did not lead to university by the reform of 1878, the B program did. Moreover, both in the A and B programs, the students were supposed to take more courses in science. In 1894, the number of lessons in mathematics and science was increased in the B program even more, while the classical languages got fewer lessons.

versities specialized in technology, economics, and agriculture. These educations were not open to students in the Humanistic program.[54]

The importance of Realistic education was accentuated by the new school regulation for the secondary schools launched in 1905. By this reform, Läroverken comprised the Realskola (lower secondary, grades 4-8[9]) and the Gymnasium (upper secondary, grades 9-12). The mere word 'Real' in Realskolan gives an indication of what the politicians had in mind. Realskolan contained no courses in classical languages. The most extensive subjects were mathematics, science, and modern languages. Moreover, Realskolan should prepare the students for vocational educations *as well as* further studies at the Gymnasium. Furthermore, there was no other program to choose from at lower secondary level. The Gymnasium, on the other hand, comprised three programs denoted Real, Classic and Half Classic.[55]

Historians that have investigated the Swedish educational system of the late 19th and early 20th centuries, such as Richardson (1963) and Florin & Johansson (1993), describe the debate on educational reforms as a part of a battle between two cultures that culminated during the two last decades of the 19th century. On one side, there were different fractions of industrialists, scientists, radical intellectuals, liberals, and socialists who demanded different types of societal changes. On the other side, there were conservative elites linked to the state administration, the church, and the state universities, who defended the current order. As I see it, Sjöstrand (1965) makes a similar description, but he does not stress the conflicts as much. He points out that people of the educated middle classes often took more moderate positions or none at all, accepting arguments from both sides.[56]

Lövheim (2006) also emphasizes that the growing importance of Realistic education in the Swedish school system did not take place without resistance and fierce debates. In his study on debates that preceded the launch of new national curricula during the period 1905-1965, Lövheim (2006) identifies different types of arguments regarding Realistic education. According to its proponents, the improvement and expansion of Realistic education was linked to the modernisation of Swedish society and future possibilities. This view also included the notion of competition between countries and an apparent risk of Sweden being left behind.[57] During the first three decades of the 20th century, the belief in the possibilities of sciences and mathematics was influential. Especially school mathematics was considered important in this context and the content of the courses was closely linked to the training of engineers at the technical universities and other technical schools, but also to research and applications in chemistry and physics. However, the Human-

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[54] Florin & Johansson (1993), p. 108
[55] SFS 1905:6
[56] Sjöstrand (1965), pp. 201-310; Richardson (1963), pp. 430-434; Florin & Johansson (1993), pp. 105-120
[57] Lövheim (2006), pp. 87-89

ists did not rest their case. In their argumentation, the negative effects of a school system too specialized in sciences and mathematics were brought to the fore; by this kind of education, the coming generations' contact with their cultural heritage together with their general education and character were at risk. Eventually, the Humanists were successful in the sense that mathematics and biology, the two dominating school subjects in Realskolan and the Realistic program in Gymnasiet, were drastically reduced by the school reform of 1933.[58]

In the debate that preceded the great school reforms of the 1960's, the issue regarding Realistic versus Humanistic education was not as salient as before. The importance of science and technology for the well-being of society was generally accepted. This time, the recruitment of future engineers and professionals in science was the basic issue for the authorities. Generously proportioned educations in these areas were supposed to secure the industrial production and economical growth. If we consider the number of students that entered a more advanced Realistic education at secondary level, the reforms did constitute an extension of Real education. On the other hand, the total number of lessons in mathematics and science at the Realistic program was slightly reduced.[59]

Lövheim (2006) stresses that secondary teachers in mathematics and science played a crucial role in the process where Realistic education gained greater importance, both in the curricula and in the public debate. He describes them as the "undergrowth of a new enlightenment" working anonymously, but purposefully, behind leading politicians and intellectuals. Lövheim (2006) also suggests that these teachers did not just want to reform the school system, they also longed for a new outlook on society, science and religion.[60] Richardson (1963) also mentions this movement of enlightenment and he characterizes it as radical; he summarizes its representatives' outlook on life as "antireligious, utilitarian, rationalistic and directed towards science"; their outlook on society is described as "anti bureaucratic and anti patriarchal motivated in terms of natural law".[61]

Integration processes in the Swedish school system

By 1880, elementary education in Sweden comprised two main types of schools: Folkskola and Läroverk, i.e. elementary schools and secondary schools. The municipal authorities administrated Folkskolan, while Läroverken were state schools or private schools. Moreover, the courses in Folkskolan did not formally constitute a preparation for later courses of Lär-

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[58] Lövheim (2006), pp. 132-134
[59] Lövheim (2006), pp. 182-183
[60] Lövheim (2006), pp. 88-89.
[61] Richardson (1963), p. 432

overken.[62] Apart from these school types, there were special schools for girls, the so-called Flickskolan. These schools were supposed to offer a female type of secondary education. A fourth type of education was home tutoring.

The education at Läroverken by the turn of the 19th century was not only an education for a few; the conditions for the teaching were significantly better as well. The teachers at Läroverken had university education, often PhDs, while the teachers in Folkskolan had a three years of teacher training based on five years in Folkskolan. Läroverken also received greater financial means. In 1870, for example, the total budget for Läroverken exceeded the total budget for Folkskolan by a factor of three, even though the number of students in Folkskolan was 50 times greater. Moreover, the sanitary conditions at the Folkskolan schools could be very poor.[63]

During the 1880's and 1890's, this organisation of the school system was intensively criticized by liberals and socialists. It was not only unfair, the critics argued, it also caused hostility between the classes since the children of the wealthy and educated and the children of the workers were kept apart. It was believed that these tensions would be harmful for society in the long run. With the intention of avoiding such tensions, reformists propagated for one type of school for all children. Another reformist argument was that the quality of the education delivered in Folkskolan had to be improved in order to elevate the education of the lower classes.[64]

During the late 19th century and the 20th century, several of the school reforms were influenced by these arguments. In stages, the Folkskolan was extended to seven years and at it was given the formal status of being a preparation for further studies at Läroverken. In 1894, it was formally settled that the three first years of Folkskolan should constitute the standard requirements for entrance to Läroverken. Folkskolan was, however, not compulsory and there were different private alternatives. When the municipal version of Realskolan, the so-called "Mellanskolan" (grades 7-10), was introduced in 1909, the importance of Folkskolan was upgraded even further. Now, the full six years in Folkskolan should provide the preparation for further education at Mellanskolan. Still, the municipal Mellanskolan was mainly established in rural areas and smaller towns without secondary schools. It was not until 1927 that the six years of Folkskolan became a preparation for further studies throughout the country. That year, Realskolan was reformed and the students could enter after the fourth or the sixth year

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[62] At the beginning of the 20th century, the schools of Läroverken were situated in larger towns and they comprised from three up to nine years. The schools of Folkskolan were situated all over the country, but they were different depending on the density of the population. In the larger towns, the schools comprised several classes with one teacher per class. In the countryside, one teacher was often responsible for two classes. In more remote parts of the country, the schools were not even stationary.
[63] Richardson (1999), p. 49
[64] Richardson (1999), pp. 68-69

of Folkskolan. Furthermore, even though Folkskolan was not compulsory, state funding for private schools was terminated in 1927, which of course made the private alternatives less attractive.[65]

The crucial step in this integration process was taken in the early 1950's, when the government decided that Folkskolan, Realskolan, and Flickskolan were to be replaced by one the compulsory school type (year 1-9), the so-called Grundskolan. The Gymnasium at Läroverken was to be replaced by the new Gymnasieskolan (grades 10-12[13]). Grundskolan was introduced in 1962 and the new Gymnasieskolan in 1964.[66] However, Grundskolan had not been prepared in a hurry. As early as 1940, the government had initiated a commission that was charged with finding a general solution to an array of pressing problems, problems that were considered not recent even in 1940. One of the pressing problems was the transfer of students between the different school types.[67] The background to this problem was the increasing number of students aiming at programs above Folkskolan

Table 1.

Year	number of graduate in Real-skolan	number of graduates in the Gymnasium
1900		968
1910	1 502	1 544
1920	2 197	2 048
1930	3 970	2 248
1940	7 005	3 839
1950	9 474	4 497
1960	27 968	9 136

In 1842, the first regulations for a compulsory elementary school, i.e. Folkskolan, were launched. By 1900, a vast majority of the children in Sweden attended Folkskolan at least five years. From about this time, the number of students at Läroverken, i.e. Realskolan and the Gymnasium, began to increase. The numbers in the table are taken from Richardson (1999).[68]

Hence, we can discern an ambition among the politicians in charge to enhance the standards of Folkskolan during the period 1905-1962. Another important group of actors were the teachers in Folkskolan. According to Englund (1994), the teachers in Folkskolan became much more organized during the first decades of the 20th century. They took control over the authoring of textbooks and a more conscious educational debate was developed. Moreover, they questioned the current order at the teacher training institutes and the influence of the church over the schools.[69] My point here is

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[65] Richardson (1999), pp. 68-70
[66] Richardson (1999), pp. 72-76
[67] Richardson (1978), pp. 11-34
[68] Richardson (1999), p. 61.
[69] Englund (1994), p. 95

that the teachers in Folkskolan were not a group of passive bystanders; they took an active part in the development of Folkskolan. As a matter of fact, some of the ministers of education during the period 1905-1962 had a background as teachers in Folkskolan.[70]

A second integration process was the admission of women to secondary schools. As I have mentioned, special schools for girls were started during 19th century, providing a sort of secondary education that could lead to university. In 1905, girls were allowed to enter some of the schools that belonged to Realskolan, but the number of schools open to both boys and girls increased. The schools that belonged to Mellanskolan, introduced in 1909, were open to girls. In 1927, as was the Gymnasium.[71]

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[70] Richardson (1999), p. 196
[71] Richardson (1999), p. 78

Figure 1.

grade												
1	2	3	4	5	6	7	8	9	10	11	12	13

Folkskolan
In 1919, Folkskolan comprised six years throughout the whole country.
In 1937, Folkskolan was officially extended to seven years, a reform that was completed only in 1949
In some municipalities, Folkskolan was extended to eight, nine or even ten years.

6-year Realskola
Introduced in 1905.

Mellanskolan
Introduced in 1909. Mellanskolan was municipal version of Realskolan.

5-year Realskola

4-year Realskola
In 1927, the first and the second year of Realskolan was dropped.
The students could chose to enter the 5-year program or the 4-year program
after four or six year in Folkskolan.

3-year Realskola
Introduced in 1957, in connection to the preperations for Grundskolan

Flickskolan, 7-year theoretical program

Flickskolan, 5-year practical program

4-year Gymnasium
Entrance after the last or the second last year at Realskolan

3-year Gymnasium

6-year Lyceum at Läroverken
Introduced in 1927, comprised Realskola and Gymnasium.
Terminated during the 1940's.

9-årig grundskola
By the school reform of 1962, Folkskolan, Flickskolan, Mellanskolan and Realskolan were replaced by Grundskolan.

3-årig gymnasieskola

4-årig gymnasieskola
Only the technical program.

By a school reform in 1964, the Gymnasium at Läroverken was replaced by the so-called Gymnasieskolan.

Normally, the students began grade 1 the year that they turned 7.

Arguments about geometry instruction in some Western countries, 1905-1962[72]

Humanistic programs and mathematics instruction

At the beginning of the 20th century, in Germany, England and the USA, the common argument as to why courses in axiomatic geometry, or any courses in pure mathematics, should be included in school mathematics was that these courses provide optimal training in reasoning. This argument prospered in contexts where secondary education programs preparing for university were supposed to provide general Humanistic education. However, the mathematics courses could be quite different among Western countries. Moreover, the notion 'training in reasoning' contained more than training in logic.

The situation in Prussia/Germany

From the early decades of the 19th century and up to the beginning of the 20th century, secondary education in Prussia was influenced by a Neo-Humanist view of education. Following Schubring (1991), the Neo-Humanists underscored the value of a general education based on classical languages and mathematics. This type of education was supposed to function as a counterweight to theological and moral dogmas as well as the specialized Realistic educations.[73] According to Jahnke (1990), a central standpoint of the Neo-Humanists was that scientific training was considered necessary for the promotion of sound reasoning. The value of knowledge based on everyday ex-

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[72] This chapter is unbalanced in the sense that I pay much attention to mathematics instruction at the secondary level and not at the common or primary level. The reason for this is that I have not found any comprehensive literature on the latter subject. Another circumstance is that the works that I refer to are written by authors that have had different points of departure. One basic difference is that geometry instruction is given considerable attention in the works on mathematics instruction in England, while algebra and analytical geometry are treated more extensively in the works on the situation in Prussia/Germany. Consequently, geometry instruction in Prussia/Germany during the 19th century is mentioned rather briefly.

[73] Schubring (1991), pp. 71-76. However, mathematics did not have the same prominent place in all educational systems of the German countries during the 19th century. Schubring (1989) links these differences to religion. At most European universities, up to end of the 19th century, the courses at the philosophical faculties, to which mathematics belonged, functioned as a preparation for further professional educations at the faculties of medicine, law or theology. Moreover, for the training in these professions, mathematics played a minor role. From the late 17th century and onwards, the philosophical faculties had a successively weaker position at the Catholic universities. Eventually, the curricula of these faculties, including mathematics, were taken over by secondary schools. At the Protestant universities, on the other hand, the philosophical faculties were assigned a greater status, which also included mathematics. One of the crucial innovations in Protestant Prussia, in this respect, was to transfer the responsibility for the training of secondary school teachers from the theological faculty to the philosophical faculty. Schubring (1989) stresses, that this reform changed the status of the philosophical faculty from just giving preparatory courses to having advanced professional programs; he also links the occurrence of mathematics as an autonomous discipline to this reform. [Schubring (1989), pp. 175-177]

periences was downgraded in this respect; that kind of knowledge was seen as less worthwhile due to tacit assumptions and imprecise concepts.[74]

At the Prussian Gymnasiums (secondary education comprising nine years), great emphasis was put on pure mathematics. Applications of any kind were scarcely treated, at least by the introduction of the new Prussian curriculum in the 1820's.[75] The major topics were algebra and algebraic analysis, while classical geometry was allotted much less time. The first four books of Euclid's *Elements* were treated only in the second year, and then together with arithmetic and algebra. The most important propositions of book 6, 11 and 12 should be treated in the third year. From the fourth year and onwards, analytical geometry replaced classical geometry.[76] According to Schubring (1991), the mathematics courses at secondary level in Prussia during the 19th century were different depending on whether you studied at the Gymnasium or the Realschule. At the so-called Realschulen, which were being established in even greater numbers by the middle of the 19th century, the courses in mathematics were more focused on applications.[77] However, Jahnke (1994) points out that the mathematics courses at the Realgymnasien and Oberrealschulen were quite similar to the mathematics courses at the Humanistic Gymnasiums.[78]

Jahnke (1990) identifies two leading figures in the efforts to develop the Prussian curricula in mathematics: August Crelle (1780-1855) and Martin Ohm (1792-1872). (It is worth noting that Crelle had a background in engineering.) Their motivation for giving pure mathematics such a prominent position was partly its training in logic. Another argument was that studies in pure mathematics imbued students with a critical mind. One component of this critical mind concerned the students' reliance on spatial intuition and their experiences of tangible reality. The other component concerned the

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[74] Jahnke (1990), pp. 14-26
[75] Jahnke (1990), p. 347.
[76] Jahnke (1990), pp. 342-351. Regarding the term algebraic analysis, Jahnke (1994) points out that fact that the advanced courses during the last five grades at the Gymnasium was based on Euler's theory on functions in the first volume of his *Introductio in analysis infinitorum*. Jahnke (1994) describes it in the following way: "Essential for Euler's conception was the algebraic view of the concept of function, and, in a natural way, this also led to an algebraic view of infinitesimal calculus. Objects and most important tools of Euler's *Introductio* were finite and infinite algebraic expressions, that is, polynomials and power series, finite and infinite products, and continued fractions as well as their transformations." [Jahnke (1994), p. 420] This Eulerian conception of analytical geometry dominated mathematics instruction at the secondary level in Prussia and the other German states throughout the 19th century.[Jahnke (1990), pp. 342-351, Jahnke (1994), pp. 426-427] It was not until the beginning of the 20th century that derivatives and integrals were included in the curriculum. [Jahnke (1994), pp. 426-427] Jahnke (1994) also uses the term geometrical analysis; as I understand it, this is when you use algebra or algebraic analysis in connection with geometry, for instance when conic sections are treated algebraically.
[77] Schubring (1991), pp. 71-84
[78] Jahnke (1994), pp. 425-426

possibilities of applying pure mathematics.[79] According to Crelle, mathematics is a rather obtuse tool since the full complexity of a problem cannot be accounted for by mathematical formulas. Knowledge in pure mathematics should therefore instill awareness of the possibilities of applying mathematics in different situations; Jahnke (1994) defines it as a general ability to orient oneself or a faculty of judgment. Moreover, Jahnke (1994) finds it relevant to talk about "indirect applications" of mathematics.[80] He puts forward the following quote by Crelle:

> Only after a mathematical *spirit* has been awakened by assiduously exercising *judgment by means of mathematics* (without regard for applications), and only then, may one quite boldly count on the uses of mathematics in applications. Mere *knowledge* of mathematics, intended for *applications* … is not sufficient for appropriate applications, but the guiding principle must be the mathematical *spirit*, the mathematical *way of thinking*. Only he who tackles applications *on this basis* will err less easily, for he will first of all examine *what* mathematics can properly achieve, and where and how the tool can be usefully applied. … Hence it is *quite right* that mathematics be exercised as much as possible in schools … at first *without* any considerations of applications in common life.[81]

The implementation of the courses in pure mathematics was not undisputed, however; Crelle even suggested that common arithmetic should not be taught at the Gymnasiums.[82] School officials, parents, and students objected to the lack of practical everyday applicability in connection with the teaching of common arithmetic. As a result, this type of everyday applications remained as a part of mathematics instruction at the Gymnasiums.[83] In contrast, the theoretical applications of arithmetic and algebra were rather limited at the Gymnasiums throughout the 19th century. The main item of the more advanced course was pure mathematics, i.e. the structural core of algebra and algebraic analysis. As soon as there were cut backs in courses, it concerned the theoretical applications, e.g. the algebraic treatment of conic sections.[84]

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[79] Jahnke (1990), pp. 19-20, 341-342. Here, Jahnke (1990) points to the influence of Kant's writings about mathematics, intuition, and reasoning. According to Jahnke (1990), Crelle and Ohm recognized Kant's claim that an a priori spatial intuition constitutes the foundation for mathematical reasoning.
[80] Jahnke (1994), pp. 419-420.
[81] Jahnke (1994), p. 419. The quote was originally printed in Crelle (1845), pp. IX-X. The translation is by Jahnke. The italics are Crelle's. Jahnke links this view to Kant and his *Critique of Pure Reason*, where Kant underscores that the application of rules and formulas in science requires an ability to make good judgments.
[82] Jahnke (1986), p. 89
[83] Jahnke (1986), p. 87; Jahnke (1994), pp. 422-424
[84] Jahnke (1994), pp. 424-426

The situation in England

Following Richards (1988) and Howson (1982), mathematics instruction at the secondary and tertiary levels was primarily considered a part of a general Humanistic education. This was the case throughout the 19th century. In England, this position within a Humanistic education was based on the widespread conception that mathematics, especially geometry, constituted the optimal subject for the training of reasoning. This kind of training should lift and ennoble the intellectual as well as the spiritual minds of the students.[85] In contrast to the situation in Prussia/Germany, classical geometry dominated the mathematics courses of the Humanistic curricula of the English grammar and public schools.[86]

Another argument in England was that geometry instruction constituted a means to attain true knowledge about the real world.[87] This view of mathematics is described by Richards (1988) as *descriptive*, meaning that true mathematical propositions must be true propositions about the real world. For instance, a definition or an axiom in geometry is inappropriate unless it expresses some essential feature of physical space. Moreover, mathematical knowledge was considered to be genuinely geometrical, while algebra and analytical geometry, due to its symbolism, was treated with mistrust in this respect.[88]

The situation in the USA

By the onset of the 20th century, training in reasoning was the leading argument for having geometry as a part of the mathematics courses at the High Schools in the USA. According to González and Herbst (2006), the main point was not to learn about geometrical concepts and ideas, but to learn, practice, and apply logical deductions. Moreover, the proponents of this

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[85] Richards (1988), pp. 14-27, 40-41; Howson (1982), pp. 88-90

[86] Jahnke (1990), pp. 334-335

[87] Richards (1988), p. 29-33. These ideas about rational reasoning and geometry instruction are explained further by Richards. This is done mainly through study of articles and essays by Whewell, Herschel, De Morgan, and Cayley.

[88] This was not just a standpoint espoused in philosophical discussions; Richards points out that De Morgan and Cayley, some of England's leading mathematicians of that time, engaged in discussions where they tried to give geometrical explanations for new results in algebraic analysis.[Richards (1988), pp. 20-39, 47, 50-55] However, Whewell's and Herschel's basic convictions regarding the nature of mathematical knowledge also maintained its differences. Whewell favored a Kantian view of this matter, arguing that geometrical knowledge was a priori our sense experiences. In England, this was a rather rare opinion during the 19th century, according to Richards (1998). Herschel, De Morgan, but also a person like John Stuart Mill and many others, argued, albeit in different ways, that geometrical knowledge is wholly empirical.[Richards (1988), pp. 25, 35] Richards especially mentions Mill's writings on mathematics and science. He very much shared the view that mathematics is descriptive in nature. Mill is interesting to Richards (1998), since he was a prominent intellectual outside the academic elite connected to the universities of Cambridge and Oxford. Hence, the ideas of Whewell, Herschel, and De Morgan concerning geometry, mathematics, and science were not restricted to groups in Oxford and Cambridge, Richards argues.[Richards (1988), pp. 34-39]

argument claimed that geometry was the ultimate school subject for this type of training. González and Herbst (2006) also point out that this argument included the view that knowledge in making logical arguments was transferable to areas outside mathematics, such as newspaper reading and democratic participation. Hence, they are here emphasizing that training in reasoning was seen as a preparation to become a functional citizen.[89] This particular concept, i.e. the functional citizen, does not occur in the literature on mathematics instruction in Prussia/Germany and England during the 19th century. My point here is that the argumentation about geometry instruction and training in reasoning were used in discussions on education and the creation of a democratic society.

Changing arguments about mathematics instruction during the early 20th century

By the turn of the 20th century, the organization of mathematics instruction at the secondary level was criticized in different quarters. Some criticized the whole Humanistic conception of mathematics instruction, while other leaned towards more moderate changes.

The situation in Germany

At the beginning of the 20th century, the German educational system went through important changes. Also this time, mathematics was a major issue. Jahnke (1994) is here pointing at some underlying factors to why the mathematics course at the secondary level was reformed.[90]

- Algebraic analysis occupied an important part of the curriculum, but it had lost its scientific significance during the second half of the 19th century.[91]
- The general conception that there is a close bond between education and theoretical science was replaced by a view where the emphasis was put on experiences of particulars.
- The increasing importance of technology undermined the position of pure mathematics.

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[89] González & Herbst (2006), pp. 13-15. González and Herbst (2006) have investigated the argumentation about geometry instruction in the USA during the 20th century. Their main sources have been articles in teachers' journals and official investigations.

[90] Jahnke (1994), pp. 426-427

[91] Probably, Jahnke is referring here to the process where set theory became a foundation for algebra and analysis. This would then have caused the loss of scientific significance of algebraic analysis, as Jahnke defines it. In a previous footnote I have described how Jahnke considers algebraic analysis as an equivalent to Euler's work *Introductio in analysis infinitorum*. In this work, a function, for instance, is defined as an algebraic expression, which we may compare with Cantor's or Dedekind's definitions, in the late 19th century, where a function was defined as a relation between two sets of numbers.

One of the leading actors was one of the most prominent mathematicians of that time, Felix Klein. According to Jahnke (1994), the reformation of the mathematics courses was a battle between two different mathematical paradigms: on one hand, those who defended the extensive courses in algebraic analysis, on the other, Klein and his followers, who wanted to insert differential and integral calculus together with geometrical applications. Schubring (1989) also stresses that one of Klein's concerns was the technical universities. A major problem to Klein was the low standard of mathematics instruction at these institutions. He linked this low standard to the mistrust between engineers and mathematicians; the former group cared for applications and disdained pure mathematics and vice versa. This was a conflict that Klein aimed to resolve. As a result of Klein's efforts, differential and integral calculus became major topics in the mathematics courses in the secondary schools, both in the Gymnasiums and the Realschulen.[92]

An important aspect of Klein's involvement in mathematics instruction was his view of intuition, rigor, and reasoning. According to Rowe (1985), Klein's view on the goal of mathematics instruction was in tune with the Neo-Humanistic conception of pure mathematics and training in reasoning.[93] Rowe (1985) backs his standpoint by the following passage in Klein's *Erlanger Antrittsrede* from 1872:

> By the word "applications" I am thinking much more of the theoretical services performed by mathematics in the development of other sciences – I am also thinking in particular of the formal educational value that the study of mathematics has. … the value of mathematics lies less in the knowledge gained through its applications, although this is certainly not to be undervalued, than through the *training of the mind* gained through working with pure mathematics. In this sense, the study of mathematics, as has long been recognized, has become more than ever a necessity for the general scientist, and especially as exact investigations are becoming more widespread in the individual disciplines. *Mathematics as a formal educational tool* – that is the key phrase which I would implore students of the sciences and medicine to bear in mind.[94]

However, Rowe (1985) suggests that Klein's idea of formal value was "quite different from the formalism that dominated German mathematics instruction". Rowe (1985) sees the following quote of Klein as a sharp criticism of the "Formalismus" at the Gymnasiums and the universities.

> Instead of developing a proper feeling for mathematical operations, or promoting a lively, intuitive grasp of geometry, the class time is spent learning mindless formalities or practicing trivial tricks that exhibit no underlying principle. One learns to reduce with virtuosity long expressions that are de-

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[92] Schubring (1989), pp. 181-192.
[93] Rowe (1985), p. 128.
[94] Rowe (1985), pp. 137-138. Translation by Rowe.

void of meaning, or to apply one's diligence to the solution of artificially constructed equations that are contrived in such a fashion that one cannot even begin to make progress unless one knows some special trick in advance. When, however, the student with this sort of training is required to develop an independent idea or answer a question that is unfamiliar to him, he lacks all trace of individual initiative.[95]

According to Rowe (1985), these remarks on mathematics instruction reflect Klein's "lifelong preference for mathematical insight rather than computational virtuosity, intuition rather than rigor; and not at least, his propensity for geometric as opposed to analytic modes of thought."[96] Hence, Klein was not abandoning the idea about mathematics instruction and training in reasoning; he was criticizing what he saw as excesses in algebraic manipulations. We can compare this to the Perry movement in England, which was more skeptical about mathematics instruction and training in reasoning.

Changing arguments about geometry instruction in England

During the 19th century and the early 20th century, the prevailing view of geometry instruction and general education in England was challenged. Following the works of Richards (1985), Howson (1982), and Price (2003), the criticism came from three directions.

1. Educators who argued that geometry instruction together with the textbooks could be made more palatable to the students.[97] In comparison to the later two groups, this group did not demand radical changes of courses or goals.
2. Leading intellectuals who refuted classical Euclidean geometry and Euclid's *Elements* as a textbook from a scientific point of view.[98]
3. Those who advocated another kind of general education. They rejected the exclusive effect of geometry instruction on reasoning and demanded that mathematics instruction, geometry included, should be adapted to practical matters.[99]

Let us first consider group number one. Beginning in the 1860's and a couple of decades onward, alternative elementary textbooks in axiomatic geometry were published. However, none of these textbooks was a success, and they were ruthlessly criticized by mathematicians, e.g. De Morgan, who had superior skills in mathematics.[100]

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[95] From Rowe (1985), p. 139. Translation by Rowe.

[96] Rowe (1985), p.

[97] Howson (1982), pp. 123-137, Richards (1988), pp. 164-185

[98] Price (2003), p. 466

[99] Howson (1982), pp. 141-163; Richards (1988), pp. 185-198; Price (2003), pp. 465-466

[100] Both Richards (1985) and Howson (1982) discuss essays and textbooks by James Wilson in particular. Richards (1985) points out that Wilson considered his book as a rigorous alternative to Euclid. Hence, it was not a textbook in applied geometry. Richards (1985) gives the following description: "Wilson abandoned the definitions and axioms on which Euclid had based his theory in favor of a totally new approach. The linchpin of Wilson's innovation was

In the second group of critics, we find intellectuals like Bertrand Russell who engaged in discussions on geometry instruction. In an article from 1902 in a periodical on mathematics instruction, *Mathematical Gazette*, Russell refuted Euclid's *Elements* as a textbook in schools. In this article, he resumed the critique regarding tacit assumptions and the lack of rigor that had been delivered during the 19th century. Due to these gaps, he found the advantages of Euclid's *Elements* highly exaggerated with respect to logic.[101]

Finally, the third group of critics, which include the so-called Perry movement, was active at the beginning of the 20th century. The leading person of this group of critics was John Perry – a mathematics teacher at a technical college. Perry and his followers rejected the usefulness of pure mathematics and its effects on reasoning. They also considered it a major problem that the final degree examinations had to adhere to the standards of pure mathematics, which in the case of geometry was equivalent to the axiomatic method. As Perry saw it, this restricted the possibilities of giving courses in applied mathematics. Perry's efforts were successful and eventually the regulations were changed during the first decade of the 20th century. Perry also advocated teaching methods where geometry and the justification of propositions were treated in an experimental fashion and not by means of the axiomatic method. Another critic was Charles Godfrey (1876-1924), whose comment on the situation in England illustrates the spirit of the Perry movement:

> In England we have a ruling class whose interests are sporting, athletic and literary. They do not know, or if they know do not realize, that this western civilisation on which they are parasitic is based on applied mathematics. This defect will lead to difficulties, it is curable and the place for curing it is school.[102]

However, this attitude did not entail that all the critics rejected axiomatic geometry. Godfrey, for instance, authored textbooks according to the axiomatic method that were to serve as an alternative to Euclid's *Elements*. I will soon return to Godfrey's textbooks below.

During the 20th century both practical geometry and the experimental approach were indeed picked up at secondary schools and colleges.[103]

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the basically indefinable concept of "direction." He introduced this notion in his fourth definition: "A *straight line* is a line which has the same direction at all parts of its length." It was elaborated somewhat further in the sixth definition: "Two straight lines that meet one another have different directions, and the difference of their direction is the *angle* between them.""[Richards (1988), p. 180]

[101] Russel (1902), pp. 165-167
[102] Howson (1982), p. 158
[103] Howson (1982), pp. 141-163; Richards (1988), pp. 185-198; Price (2003), pp. 465-466

Changing arguments about geometry instruction in the USA

During the first half of the 20th century the argument regarding geometry instruction and training in reasoning was challenged, by a line of arguments that González and Herbst (2006) denote as utilitarian. The main goal of the proponents of this argument was to match the geometry courses to the demands of the students' future jobs, which mainly was the job of a worker. It is not obvious that this goal leads to geometry courses dominated by calculations of lengths, areas and volumes and various applications, whereas the axiomatic method is cancelled. Future workers may also need training in logic. The crucial detail in the argumentation of the utilitarians was that teaching should contain apparent references to the future jobs of workers. In that perspective, practically oriented geometry courses without the axiomatic method were considered more appropriate.[104]

According to González and Herbst (2006), the 20th century contains the rise and fall of the argument regarding training in reasoning. This was the leading argument at the beginning of the century, but in an official report as early as 1909, the utilitarian argument had become established.[105]

Experiments and spatial intuition – ways to develop geometry instruction

According to Fujita, Jones & Yamamoto (2004), experimental and intuitive approaches to geometry instruction in secondary schools were discussed in Germany and England by the turn of the 20th century. Influential opinion makers in these discussions were John Perry in England and Felix Klein in Germany. In both these countries, official reports occurred that stressed the importance of such teaching methods. Moreover, these reports gained influence and experimental and intuitive approaches were included in the first geometry courses at the secondary schools.[106]

The investigation of Fujita et al (2004) shows that textbooks were produced in accordance with these guidelines regarding experiments and spatial intuition. Among these we find textbooks by Treutlein (1845-1912) in Germany and Godfrey (1876-1924) in England. Godfrey's textbooks in axiomatic geometry, intended for later years, also included exercises with an experimental and intuitive approach.[107] Fujita et al (2004) claims that these textbooks of Treutlein and Godfrey were popular among the teachers. In the case of Godfrey's alternative to Euclid's *Elements*, Howson (1982) considers this geometry textbook to have been the most popular in England by the first years of the 20th century.[108]

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[104] González & Herbst (2006), pp. 16-18
[105] González & Herbst (2006), p. 27
[106] Fujita, Jones & Yamamoto (2004), pp. 2-3
[107] Fujita, Jones & Yamamoto (2004), p. 6
[108] Howson (1982), p. 150. Godfrey's textbook was not the only alternative to Euclid's *Elements*, other did appear as well. An important detail is that the authors of the alternative text-

According to Fujita et al (2004), both Treutlein and Godfrey recognized two modes of thinking in connection with geometry:

1. logical thinking
2. an ability related to intuition

Treutlein called the later ability "das raumliche Anschauungsvermogen", which is translated to "spatial intuitive ability" by Fujita et al (2004). Godfrey called this ability the "geometrical eye", which he defined as "the power of seeing geometrical properties detach themselves from a figure". Both Treutlein and Godfrey underscored the importance of developing this kind of intuitive thinking in connection to geometry instruction.[109] Godfrey exemplified this kind of thinking in the following way:

> Experimental and intuitional methods are not identical. … Take the equality of vertically opposite angles. If I measure the angles I am proceeding experimentally; if I open out two sticks crossed in the form of an X, and say that it is obvious to me that the amount of opening is equal on the two sides, then I am using intuition.[110]

By including exercises that activated the students, the author intended for them to become aware of plane surfaces in an intuitive way. An example from one of Treutlein's textbooks is an exercise where the students are supposed to make new figures by moving triangles.[111]
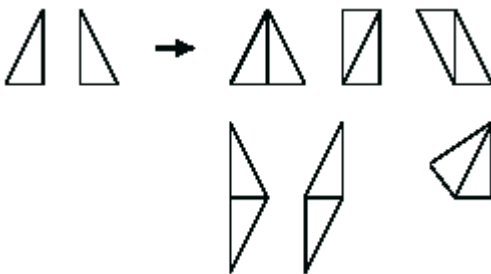
According to Howson (1982), Godfrey's design of geometry textbooks took a new turn during the early 1920's as he tried to replace proofs based on congruence by more informal ones based on symmetry. However, these textbooks did not become a success.[112]

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books did not abandon the deductive style. The introduction of alternatives to the *Elements* was not just a question of changing books; the new textbooks were also matched with changes in the requirements for the final exams at Cambridge University. [Howson (1982), pp. 150-152]

[109] Fujita, Jones & Yamamoto (2004), pp. 3-8
[110] Fujita, Jones & Yamamoto (2004), p. 6
[111] Fujita, Jones & Yamamoto (2004), p. 4
[112] Howson (1982), p. 167

A detail regarding the work by Fujita et al (2004) is that they mainly treat geometry textbooks during the first half of the 20th century, whereas they do not pay the previous century any particular attention. However, they do not say that the ideas about intuition and geometry instruction were innovations of the early 20th century. On the contrary, they suggest that both Treutlein and Godfrey were influenced by J. F. Herbart (1776-1841). As early as 1802 Herbart argued that imaginative skills are important in connection to geometry instruction.[113] Jahnke (1986) suggests that intuition and visualizations were discussed in connection to school mathematics in Prussia/Germany throughout the 19th century.[114]

An important aspect of the works of Howson (1982) and Fujita et al (2004) is that they display that experiments and intuitive approaches indeed were a part of geometry instruction during the first decades of the 20th century. These questions were indeed debated; they were taken up in reports, and textbooks were designed according to these ideas.

Apart from the development of textbooks, Treutlein was also involved in a movement that advocated a teaching method called "Kopfgeometrie" or Mental geometry. The history of this method is given by Schmidt (2002). According to this method, the students are supposed to consider geometrical objects and theorems by pure imagination; they are not supposed to use pencil and paper. By detailed descriptions, the teacher guided the students through definitions of concepts, constructions, and theorems. Schmidt (2002) links this teaching method to the pedagogical works of Pestalozzi and Herbart, where the importance of spatial intuition in connection with geometry instruction is emphasized. According to Schmidt (2002), the first text on the subject occurred in the early 19th century in today's Germany, but the method only became influential in Germany during the period 1890-1933.[115]

However, the attempts to introduce experimental and intuitive approaches to geometry did not succeed in all European countries. Toumasis (1990) suggests that Greek mathematicians and educators have felt a certain obligation to protect and maintain an original approach to geometry. Throughout the 20th century, cultural and historical arguments have been very influential in the struggle to keep Euclid's *Elements* as a textbook.[116]

New forums for mathematics instruction - L'Enseignement Mathematique and ICMI

Schubring (2003) stresses that international cooperation regarding mathematics instruction before 1900 was almost non-existent and that mathematics instruction was quite different in the Western countries.[117] This situation

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[113] Fujita, Jones & Yamamoto (2004), p. 11
[114] Jahnke (1986), p. 93
[115] Schmidt (2002), pp. 5-32
[116] Toumasis (1990), pp. 491-508
[117] Schubring (2003), pp. 54-55

began to change with the foundations of the journal *L'Enseignement Mathematique* in 1899 and the International Committee for Mathematics Instruction, ICMI, in 1908. When the ICMI was founded, *L'Enseignement Mathematique* became its official journal.[118] Anyhow, the purpose of these institutions was not just to facilitate international cooperation. The purpose was also to improve mathematics instruction. At this juncture we should recall Felix Klein, the first president of ICMI, and his involvement in the reformation of school mathematics. We should also notice Henri Fehr (1870-1954), another of the founders of the ICMI and its first general secretary. According to Furinghetti (2003), his articles in *L'Enseignement Mathematique* concerned the following topics:[119]

- innovations in mathematical programs and their links with development of science and technology
- the relationship between pure and applied mathematics and its influence on mathematics teaching
- teacher training
- new trends in mathematics teaching

One of the three British delegates at the ICMI was Charles Godfrey.

Hence, in the founding of *L'Enseignement Mathematique* and the ICMI, we see not only an international forum for discussions on mathematics instruction, but also a leadership composed by prominent mathematicians and educators with a keen interest in the reformation of mathematics instruction.

From the beginning, ICMI was supposed to deal with mathematics instruction at the secondary level, even though there was a tendency to focus on the links to university mathematics. Eventually, the work was extended to lower levels as well.[120] The most urgent topic during the first years was geometry instruction. Of the 504 articles in *L'Enseignement Mathematique* during the period 1899-1914, 150 were about geometry.[121] Furinghetti (2003) describes the articles in the following way:

> The majority of the contributions were on geometry. That this was at that time considered to be the backbone of the mathematical instruction at secondary level in many countries is shown by the many letters from readers discussing themes related to Euclidean geometry. … Often one feels that behind many articles lay the problem of answering such questions as the role of rigour and axiomatic methods in the teaching of mathematics. This subject is linked to the foundational debate, very much alive in those years. Indeed, following the birth of ICMI the debate on the place of foundations in mathematical instruction became the object of specific inquiries published in the journal.[122]

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[118] Furinghetti (2003), pp. 28-29
[119] Furinghetti (2003), pp. 25-29.
[120] Schubring (2003), p. 57
[121] Furinghetti (2003), p. 31
[122] Furinghetti (2003), p. 32

At the ICMI conference in Milan 1911, axioms and rigor in secondary schools was one of the two major themes. Also intuitive and experimental approaches in mathematics were an important issue. At the ICMI conference in Cambridge 1912, intuition and experimental evidence was one of the two major themes.[123]

However, these were not the only main issues treated within the ICMI. Others were for instance Felix Klein's other favorite issues: the extension of courses in calculus and the relationship between pure and applied mathematics. According to Schubring (2003), his efforts to extend the calculus courses at the secondary level were successful also outside Germany.[124]

Changing arguments about geometry instruction during the 1950's

As I have mentioned in Part A, some arguments were more influential than others in the Swedish debate on mathematics instruction during the 1950's and 60's. One of these arguments was based on the claim that school mathematics was old-fashioned from a scientific point of view. I would say that this argument was linked to the international discussions about mathematics instruction among educators and mathematicians.

One of the best-known expressions in the discussions on geometry instruction during the 1950's and 60's is "Euclid must go", expounded in 1959 by the French mathematician Dieudonné at a conference in Royaumont in France. Howson (2003) implies that this cry for change was not as drastic as we might think. First of all, textbooks that followed Euclid very closely were not common at the secondary level in the Western countries at this time. However, they did have an axiomatic structure and the central theorems where the same as Euclid's. Secondly, other mathematicians well before Dieudonné, like D.E. Smith in 1911, had expressed similar thoughts.[125]

Dieudonné's actual proposal for what geometry instruction should include and how it should be organized is described by Howson (2003) in the following way:

> Dieudonné himself recommended that up to the age of 14 the teaching of geometry should be "experimental" and "part of physics, so to speak". But the

[123] Furinghetti (2003), p. 43
[124] Schubring (2003), pp. 60-63
[125] Howson (2003), p. 116. Howson gives the following quote from Smith: "The efforts usually made to improve the spirit of Euclid are trivial [...] but there is a possibility [...] that a geometry will be developed that will be as serious as Euclid's and as effective in the education of the thinking individual. If so, it seems probable that it will not be based upon the congruence of triangles, but upon certain postulates of motion [...]. It will be through some efforts as this, rather than through the weakening of the Euclid-Legendre style of geometry, that any improvement is likely to come." [Howson (2003), p. 118. Originally, Smith (1911)] The brackets in the quote are placed by Howson's.

emphasis should not be on "such artificial playthings as triangles" but "on basic notions such as symmetries, translations, composition of transformations, etc". An argument that in retrospect does not look terribly revolutionary. However, from the age 15 he proposed the introduction of the axiomatic method: "The axioms should be developed from the algebraic and geometric point of view, i.e. any notion should be given with both kinds of interpretation. […] the emphasis should be on linear transformations, their various types and the groups they form. Matrices and determinants of order 2 appear […] in a natural way in this development." [Dieudonné 1961][126]

The importance of Dieudonné's call for a reformation of geometry instruction was not so much its actual suggestions; the changes that followed in several countries were far from consistent in following these suggestions. Dieudonné's talk was more of an introduction to various activities. One effect was that several other prominent mathematicians produced articles about what school geometry should include and how it should be taught. Yet the vast majority of these suggestions remained were never realized.[127] A second effect was that the content of the geometry courses indeed was radically reformed in several countries. Howson (2003) distinguishes two main paths for textbook and curricular development.

1. To keep a thorough going axiomatic structure, but replacing the Euclidean axioms with concepts and axioms from linear algebra, vector geometry or topology.[128]
2. To drop a thorough going axiomatic structure, but having "small packets of deductive geometry".[129]

Howson (2003) is quite critical about these reforms and he considers them a major failure. One reason for this failure, he suggests, was that the initiators of the reforms were mathematicians with very little or no experience of teaching adolescents.[130] Another reason was that radically different courses were supposed to be implemented during a time period that was too short. There was simply no time for the necessary preparations.[131] Fey (1978) points out that there were critics during the 60's and 70's who found the New Math an excess in abstraction and symbolism.[132]

González and Herbst (2006) describe the American discussions regarding geometry during the 1960's somewhat different. A leading argument in the USA was that the students were supposed to experience the practice of being a mathematician, which included both the heuristic aspect as well as the proofs. This line of arguments included different notions about what the

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[126] Howson (2003), p. 118. The brackets are Howson's.
[127] Howson (2003), p. 119
[128] Howson (2003), pp. 118-123
[129] Howson (2003), p. 123. Howson does not describe which axioms or theorems were used in these packets.
[130] Howson (2003), pp. 118
[131] Howson (2003). pp. 127-129
[132] Fey (1978), p. 345

courses should include; some debaters argued that non-Euclidean geometries would be more suitable than classical Euclidean geometry; others suggested linear algebra; some thought that problem-solving should be the main feature.[133]

A summary of arguments about geometry instruction in the Western countries, 1905-1962

The basic argument for having courses in axiomatic geometry, or other courses in pure mathematics, has been that they provide excellent training in reasoning. In Prussia/Germany and England during the 19th century, this type of argumentation was closely linked to the goal of the secondary schools to provide general Humanistic education. In this respect, algebra was the dominating subject in Prussia/Germany; in England it was geometry. However, the notion of 'training in reasoning' encompassed more than logical thinking. Studies in pure mathematics were also supposed to cultivate the students' character. Furthermore, in England, studies in geometry were considered to provide knowledge about real space. In Prussia/Germany, studies in algebra were considered to develop a critical awareness about spatial intuitions, but also the possibilities of applied mathematics. Hence, in the justifications for having courses in pure mathematics at secondary level, there were clear connections to studies of things and nature. Consequently, it is not possible to uphold a clear distinction between the arguments about mathematics instruction and Humanistic and Realistic educations in the sense that the former type of education was focused merely on reasoning, while the latter type was focused merely on things and nature. Nonetheless, one should keep in mind that Realistic educations at the secondary level that contained courses in applied mathematics existed in the 19th century.

In the first decades of the 20th century, mathematics instruction in Germany was reformed; at secondary level, analytical geometry with differential and integral calculus became an important part of the courses. Yet the arguments about pure mathematics and training in reasoning remained vital, for instance through Felix Klein. Also in the USA, about this time, similar arguments about training in reasoning were influential. But, here we can also observe that the training in reasoning was linked to the aim of educating independent citizens for a democratic society.

In England, the prevailing organization of geometry instruction was criticized from two directions. One group of critics considered Euclid's *Elements* inappropriate for training in reasoning; they were then referring to the logical gaps in Euclid's axiomatic system that had been discovered during the 20th century. Another group of critics, the so-called Perry Movement, argued that axiomatic geometry had too much influence over the exams at the secondary

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[133] González and Herbst (2006), pp. 18-20

level. Moreover, they considered the positive effects on the students' abilities to reason to be highly exaggerated. Instead they advocated more practical applications. Also in the USA, arguments similar to those of the Perry Movement gained influence.

However, it was not only the aims and content of geometry instruction that were discussed. The teaching methods and textbook design were also discussed during the early decades of the 20th century. Spatial intuition and experimental teaching methods were important concepts in the discussions on how to develop geometry instruction. These concepts did in fact gain influence in the design of textbooks and curricula.

Another type of change that took place during the first decades of the 20th century concerned the opportunities of exchanging experiences and ideas internationally. Such contacts became much simpler via periodicals, reports, and conferences organized by the ICMI; this organization was founded in 1908.

During the 1950' and 60's, the mathematical fundaments of the geometry courses were being questioned, primarily by mathematicians interested in education. The development of textbooks and curriculum was then focused on how to replace axioms and theorems originating from Euclid's *Elements* with new axioms and theorems from linear algebra, vector geometry, or topology.

Part C – Professional debates about elementary geometry instruction

Introduction

In this part of the thesis, I describe the main arguments regarding geometry instruction in the specialized literature on mathematics intended for teachers in Folkskolan and Realskolan. Apart from the arguments in articles and essays, I also describe the directives in the curricula.

Sources

If we mean by a debate two or more debaters exchanging arguments in a journal, all sources that are used for this chapter were not a part of a debate. The chapter on geometry instruction in Folkskolan is primarily based on three essays that were used in teacher training. However, they were not plain teaching manuals; the authors brought up basic and general standpoints regarding mathematics instruction, and they provided arguments for why mathematics should be taught in a certain way. In this respect, the arguments that I describe in this chapter were part of a debate, or perhaps an introduction to a debate. As I see it, we can very well consider them as an incentive for actions and a source of arguments.

Originally, my idea was to investigate articles on geometry instruction in journals intended for teachers in Folkskolan. However, the outcome of my search for such articles in one of the leading journals was quite meager. Having browsed the editions between 1921 and 1938 of *Folkskollärarnas tidning*, one of the major journals, textbook reviews were the most exhaustive articles on geometry instruction that I found. But the reviews are also a poor source; all reviews were kept in a neutral tone and the authors refrained from delivering any real criticisms; the authors did not really explicate any reasons for why they thought a textbook was good or bad. Therefore, I gave up the search for more exhaustive articles in other journals and I turned to treatises used in teacher training instead.

In comparison, the supply of texts about geometry instruction in Realskolan was different. The first complete work on mathematics instruction in

Realskolan appeared in 1956.[134] This work was authored Halfrid Stenmark and it was named Matematikundervisningen i realskolan och motsvarande skolformer (~Mathematics instruction in Realskolan and corresponding school forms). The style of Stenmark is quite different from the treatises intended for Folkskolan. In the latter works, the authors established some basic methodological principles from which they derived teaching advice. Stenmark (1956), on the other hand, refrained from giving any type of basic principles. His work is more of a condensed, but still detailed, description of how to teach each part of the mathematics courses in Realskolan from the beginning to the end. Since Stenmark (1956) did not provide any arguments to why he gave certain advice, his work has not been one of my main sources.

Instead, the most exhaustive articles on geometry instruction in Realskolan occurred in the periodical *Elementa*[135], but on some occasions also in the journals *Tidskrift för skolmatematik*, *Tidning för Sveriges läroverk*, *Pedagogisk debatt* and *Pedagogisk tidskrift*. Of these journals, *Elementa* and *Tidskrift för skolmatematik* were specialized in mathematics instruction. In *Elementa*, physics and chemistry was treated as well.

It is worth noticing that mathematics instruction in Folkskolan, on one hand, and in Realskolan, on the other, seems to have been discussed in different types of media. If mathematics instruction was not an issue in the other journals intended for Folkskolan, it says something about the status of the subject. Another aspect is then how the arguments about mathematics instruction were conveyed to the teachers. In the case of Folkskolan, the literature used in teacher training conveyed a set of principles from which recommendations for teaching were derived. An important aspect of these works is that the basic principles were never called in question; the arguments of the authors went unchallenged. In the case of Realskolan, the basic principles were presented in journals. Here, they were discussed, but also criticized. In this way, the teachers in Realskolan could get acquainted with different standpoints on mathematics instruction.

Certainly, it would be interesting to go through all journals intended for the teachers in Folkskolan and investigate whether or not mathematics instruction indeed was a neglected issue. Still, even though such an investigation would provide a nice backdrop to my investigations, I think that it falls outside the purpose of this thesis. Such a project takes aim at mathematics instruction in general and not geometry instruction in particular.

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[134] Stenmark (1956). At any rate, it is stated on the back of the book that this is the first work on mathematics instruction in Realskolan. As I see it, there is no reason to doubt this statement.

[135] The periodical was founded in 1917 and its original name was *Tidskrift för matematik, fysik och kemi*, i.e. *Journal for mathematics, physics and chemistry*. In 1938 the name was changed to *Elementa*. In this treatise I use the name *Elementa* for the sake of brevity.

The professional status of the debaters

Regarding the essays on geometry instruction in Folkskolan, it has been possible determine the professional status of two of the authors: Gösta Setterberg (1870-) and Frits Wigforss (1886-1953). Both worked as lecturers at teacher training institutes, Wigforss in Kalmar and Setterberg in Falun. Eventually, Wigforss' became one of the leading actors in issues concerning mathematics instruction during the 1930's, 1940's and 1950's. Apart from being the author of various textbooks in mathematics and Swedish, he was a member of various official commissions regarding the school system; he designed the first curricula that were used in the test schools preceding the introduction of Grundskolan;[136] and he engaged in pedagogical and psychological investigations of children and learning in mathematics and language.

The authors of the articles on geometry instruction in Realskolan were: Adolf Meyer (1860-1925), Henrik Petrini (1863-1957), Johan Samuel Hedström (1876-1942), Hjalmar Olson (1884-1963), Ragnar Nyhlén (1892-1949) and Carl-Erik Sjöstedt (1900-1979). The common feature of these authors is that they had a PhD in mathematics. Moreover, all of them worked or had worked as secondary school teachers. They were also authors of textbooks in mathematics, which in most cases were intended for the secondary schools.

Apart from that, Meyer, Hedström and Olson were co-editors of the journal *Elementa*. Hedström and Olson were also involved in teacher training. In what way they were involved, I am not certain. However, they reveal that their articles on geometry instruction are based on lectures given to student teachers.

In comparison to the other authors, Petrini and Sjöstedt authored several works that were not textbooks. According to the library catalogues, Sjöstedt published works on mathematics, science, philosophy, and education. Petrini had perhaps an even wider scope of interest as he published works on mathematics, science, religion, language, and education.

During 1940's and 1950's, Sjöstedt had a high-ranking position at the national school board, i.e. Skolöverstyrelsen.

If we consider the literature on the history of education in Sweden, Petrini is the most renowned of all the authors. Here, he is described as a person who accentuated the value of education in science, while he sharply criticized programs in the humanities and religion, which were considered old-fashioned and dogmatic. Petrini is one of the secondary school teachers that Lövheim (2006) refers to when he writes about a "new enlightenment" in the early 20th century.[137]

Ragnar Nyhlén (1892-1949) is the most anonymous of the authors. Apart from the fact that he had a PhD in mathematics and that he wrote textbooks,

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[136] Wigforss & Roman (1951), Wigforss (1952)
[137] See for instance Sjöstrand (1965) and Lövheim (2006)

I have not been able to track any more data about him. Actually, I am not certain about if he worked as a secondary school teacher or not.

Taken together, all the authors of the texts that constitute the main sources for my investigation of the debate on geometry instruction belonged to some kind of elite. They were all textbook authors; some were active in teacher training; some were editors of journals; others had central positions with the school authorities. Consequently, the arguments on geometry instruction that I describe in this thesis are the arguments of an elite.

This fact leads to some consequences as I want to consider the relevance of the arguments. On one hand, we can infer that the arguments were relevant to the common teachers in the sense that the arguments were conveyed by important persons; it was not ordinary school teachers who were expressing their thoughts about geometry instruction. On the other hand, we cannot infer that the debate took aim at all the daily matters of the common teachers. Thus, when we consider the professional debate as an incentive for actions and a source of arguments, it might be that the debate treated parts of geometry instruction.

Curricula and time plans, 1905-1962

The mathematics courses in Folkskolan

During the period 1905-1962, Folkskolan had three different curricula.[138]

<div align="center">1900 => 1919 => 1955 =>1962</div>

Before the curriculum for Folkskolan in 1919, geometry and arithmetic constituted two separate school subjects. However, after 1919, the course was still denoted geometry and arithmetic, not mathematics.[139] Throughout the period 1905-1962, the core of the mathematics courses contained the four basic rules of arithmetic applied to whole numbers and fractions. Various practical applications were also a part of the courses. The courses in geome-

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[138] The following description of the courses and the directives regarding teaching methods are based on the curricula of 1900, 1919 and 1955. In the list of references, they are denoted *Normalplan för undervisningen i folkskolor och små skolor af kongl. maj:t i nåder godkänd den 7 december 1900*, Kungl. Skolöverstyrelsen (1919) and Kungl. Skolöverstyrelsen (1955b).

[139] In 1897, arithmetic and geometry constituted two separate subjects, and students got grades in both. In 1919, arithmetic and geometry became one subject; however, the subject was still denoted arithmetic and geometry and there were specific textbooks in arithmetic and geometry for some years. During the 1930's, textbooks that comprised both arithmetic and geometry began to occur.

try comprised drawings, descriptions, and measuring of straight lines and elementary plane surfaces and solids.[140]

The 1900 curriculum is brief in comparison to the later ones. It mainly established what should be taught, when it should be taught, and to what extent. No specific goals for mathematics instruction were explicated. In 1919, the main goal of Folkskolan's courses in mathematics was to prepare the students to use mathematics in daily life matters.

> The purpose of the instruction in arithmetic and geometry in Folkskolan is to convey to the children, in accordance with their age and progress, knowledge of and skills in calculations with special regards for the needs of daily life along with some familiarity with the constructions, descriptions, and calculations of geometrical magnitudes.[141]

A second goal concerned moral and economic matters; the applied exercises should be chosen in such a way that they supported good economic morals in personal and household matters.[142]

In 1955, the explicit main goal did not contain a formulation about the needs of daily life. This time, the main goal was to cultivate the students with respect to orderliness of thought, thoroughness, and persistence.[143] However, in the description of the courses, the practical applications were still important.[144] Moreover, economic morality and edification were also mentioned as a goal.[145]

Throughout years 1905-1962, the geometry course began in the fourth grade. The students were supposed to learn how to draw, describe and meas-

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[140] *Normalplan för undervisningen i folkskolor och små skolor af kongl. maj:t i nåder godkänd den 7 december 1900*, pp. 28-34; Kungl. Folkskoleöverstyrelsen (1919), pp. 59-69 Kungl. Skolöverstyrelsen (1955b), pp. 123-124. In comparison with the curriculum of 1900, the new curriculum of 1919 was more extensive and it contained more detailed descriptions of goals, teaching methods and courses. The curriculum of 1919 remained in force until 1955, but even though new editions were published during this time; those parts that concerned what I denote as content, methodology and goals of school mathematics were not altered. (See for instance the editions published in 1922, 1924, 1950 and 1952.)
[141] Kungl. Skolöverstyrelsen (1919), p. 58: "Undervisningen i räkning och geometri i folksko-lan har till uppgift att bibringa barnen en efter deras ålder och utveckling avpassad insikt och färdighet i räkning med särskild hänsyn till vad som erfordras i det dagliga livet ävensom någon förtrogenhet med geometriska storheters uppritning, beskrivning och beräkning."
[142] "Vid val av sakuppgifter bör tillses, att även sådana uppgifter medtagas, som äro ägnade att fästa lärjungarnas uppmärksamhet på hemmets ekonomi samt på sparsamhetens betydelse för den enskilde och för hemmet. Likaledes bör uppfostran till sparsamhet vara en viktig synpunkt vid uppgörandet av planen för undervisningen i bokföring." (Kungl. Skolöverstyrelsen (1919), p. 69)
[143] Kungl. Skolöverstyrelsen (1955b), p. 123
[144] Kungl. Skolöverstyrelsen (1955b), pp. 123-124
[145] Kungl. Skolöverstyrelsen (1955b), p. 127

ure "geometrical magnitudes". The advised content of the geometry courses in the curriculum of 1919 was the following:[146]

> Grade 4 The comprehension and application of measures of surfaces and spaces and the calculation of surfaces of squares and other rectangles along with spaces of the cube and other right-angled solids.

> Grade 5 A geometry course comprising lines, angles, parallelograms and triangles along with those solids, which have the aforementioned surfaces as bases and perpendicular sides against the bases, and essentially the construction, the description and the measurement of the aforesaid magnitudes along with basic practical calculations.

> Grade 6 A geometry course, comprising, aside of the previously treated surfaces and solids, those quadrilaterals yet not treated along with other polygons and circles together with those solids that have the aforementioned surfaces as bases and perpendicular sides against the bases, and essentially the constructions, the description and the measurement of the aforesaid magnitudes along with basic practical calculations.

> Grade 7 A geometry course, comprising, besides previously treated magnitudes, something about ellipses, pyramids, cones and spheres and essentially the construction, the description and the measurement of the aforesaid magnitudes along with basic practical calculations. Simple exercises in making graphs. Simple field measuring exercises.

In comparison to the 1900 curriculum , this description is a bit more comprehensive, but the basic items are the same.[147] The new curriculum in 1955 did not bring any major changes regarding the mathematics courses of the first seven years, but as Folkskolan was extended to eight or nine years, square roots and basic first degree equations with one unknown together with applications were added. The geometry courses in grades eight and nine contained the new items congruency and uniformity, but also some basic proofs, along with the older items.[148]

In the curricula of 1919 and 1955 it was established that *åskådlighet*, i.e. spatial intuition, was the leading methodological principle in connection to mathematics instruction.[149] In both curricula, the actual directives about *åskådlighet* were positioned in the very first paragraph of the so-called "guidelines" in the chapter on mathematics teaching. Here is the formulation of 1919, which was quite similar to the one of 1955.[150]

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[146] Kungl. Skolöverstyrelsen (1919), p. 60. This content is kept through the succeeding editions of the curriculum of 1919. See for instance the editions of 1922, 1924, 1950 and 1952.
[147] *Normalplan för undervisningen i folkskolor och små skolor af kongl. maj:t i nåder godkänd den 7 december 1900*, pp. 33-34
[148] Kungl. Skolöverstyrelsen (1955b), pp. 123-124
[149] Kungl. Skolöverstyrelsen (1919), p. 67, Kungl. Skolöverstyrelsen (1955b), p. 124
[150] Kungl. Skolöverstyrelsen (1955b), p. 124

> 1. *Visualizability* [~åskådlighet] should as far as possible be aimed at during [mathematics] instruction. Measuring and weighing should for instance be considered the foundation for the calculation of measures and weights, and the operations of calculation should, when possible, be made visual [åskådliggöras ~ to make something concrete] by counting objects.[151]

This methodological principle was explained more in detail by Fritz Wigforss, for one. His standpoints I this matter are treated in a subsequent chapter.

In the methodological directives on geometry instruction, there were no explicit formulations regarding spatial intuition. Still, we may conclude that all the recommendations about drawings and constructions are related to some idea about spatial intuition. This is the formulation in the 1919 curriculum, where the importance of students making measurements and constructions were underscored.

> During *geometry teaching*, the students ought to, as often as possible, execute the necessary measurements. In connection with calculations of surfaces[[152]], the students should draw the geometrical magnitude referred to and in the drawing indicate the given measures. Furthermore, geometry teaching should be supported by modeling, clipping and folding. …
>
> In connection to geometry teaching, the students ought to execute the simple geometrical constructions along with surface spreading, which are most useful in practical life.
>
> During field measuring exercises, right lines could be drawn by a string or they could be staked out by sticks. Right angles could be set off by using a cross table.[153]

On the other hand, elementary geometry instruction without drawings and constructions is hard to imagine; these formulations about geometry instruc-

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[151] Kungl. Skolöverstyrelsen (1919), p. 67: "1. Vid undervisningen bör så långt som möjligt *åskådlighet* eftersträvas. Så t. ex. böra mätningar och vägningar läggas till grund för räkningen med mått- och viktsorter, och räkneoperationerna böra, då så lämpligen kan ske, åskådliggöras genom räkning med föremål." [The italics is done by the original author.]

[152] Notice that the expression "calculations of surfaces" was used and not "calculations of areas". The former expression was by far the more common in the textbooks in Folkskolan and Realskolan during the period 1905-1965. The word "area" was quite uncommon.

[153] Kungl. Skolöverstyrelsen (1919), p. 69: "Vid *undervisningen i geometri* böra lärjungarna själva i så stor utsträckning som möjligt få verkställa behövliga mätningar. Vid beräkningar av ytor böra de tillhållas att före uträkningen upprita de ifrågavarande geometriska storheten och på ritningen utsätta de givna måtten. Undervisningen i geometri bör dessutom stödjas av modellering, utklippning och vikning. ... I samband med undervisningen i geometri böra lärjungarna få utföra de enkla geometriska konstruktionerna samt ytutberedningar, som vanligast komma till användning inom det praktiska livet. Vid fältmätningsövningarna kunna lämpligen räta linjer uppdragas med snöre eller utstakas med stavar samt räta vinklar avsättas med tillhjälp av korstavla." The italics are original.

tion may therefore not be a result of any specific pedagogical considerations regarding spatial intuition.

In the 1900 curriculum, spatial intuition did not have the same prominent position; the word was used on only twice in connection to the methodological advice when geometrical concepts, but also different measures, were to be introduced.[154] We find similar formulations in the 1889 curriculum, but not in that of 1878.[155]

An interesting detail is that girls in Folkskolan could choose not to study geometry until 1919. By that curriculum, all parts of the mathematics course became compulsory for both girls and boys. In the curricula of 1899 and 1900, geometry instruction was voluntary for girls and compulsory for boys, whereas arithmetic was compulsory for both girls and boys.[156] Before the curriculum of 1889, girls were not allowed to study geometry at all.[157]

The mathematics courses in Realskolan

In this description of the mathematics courses in Realskolan, Realskolan comprises six grades. This is not completely accurate since Realskolan was shortened in 1928. From this year, there were one four- and one five-year program in Realskolan, see the diagram N in chapter N on the Swedish school system. This reduction was done from below, i.e. the first and second grades were dropped. In 1950, a three-year program was added as well.

Originally, Realskolan comprised one type of program and it contained one program of courses in mathematics. By the school year 1934/35, three practical programs were added and the original program was termed the general program, i.e. allmän linje. The students could choose the practical programs for the last two years in Realskolan.[158] To my knowledge, all programs had the same courses in mathematics and the exam test were the same.[159] It was not until 1951 that each program was given specific courses in mathematics. From 1951, the students could choose between a shorter and

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[154] *Normalplan för undervisningen i folkskolor och små skolor af kongl. maj:t i nåder godkänd den 7 december 1900*, pp. 32, 34

[155] *Normalplan för undervisningen i folkskolor och småskolor, utgifven år 1889*, p. 29; *Normalplan för undervisningen i folkskolor och småskolor, utgifven år 1878*, p. 21-23

[156] *Normalplan för undervisningen i folkskolor och småskolor, utgifven år 1889*, p. 29; *Normalplan för undervisningen i folkskolor och små skolor af kongl. maj:t i nåder godkänd den 7 december 1900*, p. 34

[157] *Normalplan för undervisningen i folkskolor och småskolor, utgifven år 1878*, pp. 63-64.

[158] Examensutredningen (1961), p. 11

[159] I have not found any explicit official directives that establish that the different programs contained the same courses in mathematics. However, according to the school regulations of 1945, all practical programs in Realskolan should provide the same admission to higher schools as the common program plus specialized vocational educations in technology, trade/economics and domestics. (SFS 1945:581, pp. 1199, 1228) Moreover, some school subjects in the practical programs, e.g. German, English, and French, contained special applications, but not mathematics. (SFS 1945:581, pp. 1199)

a longer course in mathematics. In the subsequent description, the longer course in the common program is considered.

Realskolan got its very first curriculum in 1905. Until 1962, when Grundskolan was introduced, three more curricula were introduced.[160]

$$1905 => 1928 => 1933 => 1955 (=> 1962)$$

Throughout the period 1905-1962, mathematics was one school subject, and students received one grade in mathematics, i.e. there was no distinction between geometry and arithmetic/algebra. However, there were specific textbooks for arithmetic, algebra and geometry throughout the period.

Between 1905 and 1962, the goals of mathematics instruction stated in the curricula were quite similar, but there were also some differences. In all curricula, it was stipulated that mathematics instruction in Realskolan was supposed to convey knowledge and skills relevant to daily life. In 1905 and 1955, this goal was complemented with a goal about some general civic education. No specific applications in science or technology were mentioned in any curriculum. This is the formulation of 1905.

> The task of mathematics instruction in Realskolan is to communicate knowledge of and skills in arithmetic to the students, especially applied tasks from the daily life, along with knowledge of the elementary concepts and methods of geometry, to such extent that it corresponds to the requirements of a general civic education; at the same time, it shall constitute a sufficient preparation for those educational institutions that are linked to Realskolan.[161]

Following this quote, one may even get the impression that applications in daily life were linked primarily to arithmetic, whereas studies in geometry, together with arithmetic, was associated with the promotion of a general civic education.

In the curricula of 1928 and 1933 the formulation regarding general civic education was dropped in the paragraph on the goals of the mathematics courses.[162] However, the expression 'general civic education' was still a part in one of the initial paragraphs on the goals of Realskolan.[163] In the curricula of 1928 and 1933, the sole goal was to provide knowledge suitable for the

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[160] The following description of the mathematics courses are based on SFS 1906:10; Bergqvist, B. J:son; Wallin, Harald (1928); Grimlund & Wallin (1933) and Skolöverstyrelsen (1955a)

[161] SFS 1906:10, p. 25: "Undervisningen i matematik i realskolan har till uppgift att bibringa lärjungarna insikt och färdighet i räkning särskildt med tillämpning på uppgifter från det dagliga lifvet äfvensom förtrogenhet med geometriens elementära begrepp och metoder, allt till en omfattning, som motsvarar fordringarna på allmän medborgerlig bildning och på samma gång utgör en tillräcklig förberedelse för de fortbildningsanstalter, som ansluta sig till realskolan."

[162] Grimlund & Wallin (1939), p. 298. Bergqvist, B. J:son; Wallin, Harald (1928), p. 226

[163] Grimlund & Wallin (1933), p. 2

so-called "practical life". Whether or not geometry was included in this aim is not perfectly clear. This is the formulation of 1933, which was almost identical to the formulation of 1928.[164]

> The aim of geometry instruction in Realskolan is to convey to the students, on the basis of what is acquired in Folkskolan, insights and skills in arithmetic, especially applications from daily life, and to make them familiar with the elementary concepts and methods of geometry. [165]

In new curriculum of 1955, geometry instruction was given an explicit aim again.

> The systematic instruction in geometry shall not only convey a certain amount of knowledge to the students. Its purpose is also to give them, in accordance with their age, a suitable conception of the logical edifice of geometry. The proof methods of geometry shall also contribute to the fostering of the students' orderliness of thought and intellectual honesty and develop their ability to express their thoughts in clear and concise manner.[166]

As we can see in this passage, the goal of geometry instruction was not only to convey knowledge in geometry, nor just skill in logic; the formulations about intellectual honesty establish that geometry instruction was supposed to provide a kind of moral schooling as well. Even though we cannot find similar passages in the previous curricula, these directives were by no means novel in 1955. In teacher journals and textbooks, similar arguments had been conveyed at least since the beginning of the 20th century. I describe these arguments in a chapter below.

   As I see it, the goals regarding training in reasoning and intellectual honesty were a more detailed version of the formulations about providing a general civic education, which occurred in the curricula of 1905, 1928 and 1933. Both reasoning and intellectual honesty are skills and morals that are not bound to specific school subjects, professions or academic disciplines; in that sense, they are general. Moreover, these goals take aim at the student's future capacity to engage in intellectual discussions, i.e. his or hers future role as a citizen.

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[164] Bergqvist, B. J:son; Wallin, Harald (1928), p. 226
[165] Grimlund & Wallin (1933), p. 298: "Undervisningen i matematik i realskolan har till uppgift att, på grundval av vad som inhämtats i folkskolan, bibringa lärjungarna insikt och färdighet i räkning, särskilt med tillämpning på uppgifter ur det praktiska livet, samt att göra dem förtrogna med geometrins elementära begrepp och metoder."
[166] Skolöverstyrelsen (1955a), p. 118: "Den systematiska geometriundervisningen skall icke blott bibringa lärjungarna ett visst kunskapsmått. Den har också till ändamål att ge dem en efter deras åldersstadium avpassad uppfattning om geometrins logiska uppbyggnad. Den geometriska bevisföringen skall också bidraga till lärjungarnas fostran till tankereda och intellektuella ärlighet och utveckla deras förmåga att klart och koncist uttrycka sina tankar."

The prescribed content of the mathematics courses in Realskolan was very much the same throughout the period 1905-1962. The mathematics courses comprised three topics: arithmetic, algebra, and geometry. None of the courses included analysis/calculus, trigonometry or statistics.

During the first three years in Realskolan, the courses were dedicated to arithmetic. The courses comprised whole numbers, decimal numbers, and fractions together with percentage. A specific item in the curricula was mental arithmetic. At the end of the third year, the students were introduced to parentheses and the concepts *term* and *factor*.

In the fourth year, the students were introduced to equations and algebra. At first, the students were supposed to work with simple equations of the following type:

Solve the following equations (through arithmetical reasoning):

1. a) *2x+3.27=8.53*; b) *3x+6.21=16.23*; c) *5x+19.1=2.6 …*[167]

Arithmetical reasoning is not defined in the curricula, but by later passages it is becomes clear that it is something different than applying algebraic rules and methods.

As the equations became more complicated, the students should be introduced to algebraic rules and methods, which comprised reduction of terms, elimination of parentheses, insertion of factors within parentheses, multiplication of simple polynomials and factorization of simple polynomials, in this case second degree polynomials. During the fourth and the fifth year, the students worked with linear equations with one unknown. In the sixth year, the students should learn how to solve systems of two linear equations with two unknowns. However, the courses did not include second-degree equations. Neither did the courses in Realskolan comprise polynomial division and factorization of more complicated polynomials. Here are a couple of equations from the final exams of Realskolan in 1925 and 1946.[168]

$$\frac{x+\frac{1}{4}}{x-\frac{1}{6}} - \frac{x-\frac{1}{5}}{x+\frac{1}{2}} = \frac{47}{60x-10} \qquad (1925)$$

$$\frac{7}{3-3x} - \frac{4x+19}{3-3x^2} + \frac{x}{1-2x+x^2} = 0 \qquad (1946)$$

[167] Rendahl, Wahlström & Frank (1940), p. 5
[168] Stenbäck & Sundbäck (1962), pp. 10, 13. The first equation was the first task out of eight at the final exam in mathematics of 1925. The second equation was the first task out of eight at the final exam in mathematics of 1946. In 1925, 3785 students took the test in mathematics, 2268 of these tried to solve the equation and 1759 passed. In 1946, 8239 students took the test in mathematics, 7033 of these tried to solve the equation and 6118 passed.

The so-called "applied tasks from daily life" comprised exercises within the following topics: rule of three, interest rates, discounts, mean values, bookkeeping, mixtures, and uniform motions. Arithmetic and algebra was also used to compute lengths, areas and volumes. In connection with these computations of lengths, areas, and volumes, the students were introduced to irrational numbers such as π and square roots.

Geometry instruction began in the third year of Realskolan. During this year, the students should be introduced to the basic concepts: straight line, plane surface, angles, different types of straight lined surfaces, circles, sectors, etc. However, the approach was supposed to be less formal, i.e. the concepts should not be introduced by means of explicit definitions and theorems; instead the students should learn about the geometrical concepts as they worked with various applied exercises. The exercises were supposed to involve some kind of measuring, e.g. an investigation of a diagram by means of a ruler or a protractor. In addition, larger objects such as the classroom or a field could be investigated. In connection with these measuring exercises, the students were supposed to be acquainted with the concepts of proportion and uniformity as well. During the third year, the students should also be introduced to basic concepts in stereometry, e.g. prism, cylinder, pyramid, cone, and sphere.

Thus, the first geometry courses in Realskolan were quite similar to the geometry courses in Folkskolan.

In the fourth year, the students were introduced to axiomatic geometry, but the students were to continue to learn about measuring and applied geometry. Axiomatic geometry was, in its main parts, taught in the fourth and fifth year. The reason why axiomatic geometry was not taught in the final sixth year was probably that most of the students that went on to the gymnasium did so after the fifth year.

If we consider the curricula, they do not reveal much about the content of the courses in axiomatics. A much better source in this respect is the textbooks. Throughout the period 1905-1962, they contain a core of theorems that in large parts correspond to the theorems in books I and III of Euclid's *Elements* together with some theorems on uniformity and proportions. If we consider the geometry exercises at final exams in mathematics and the requirements for their solutions, we can confine this core even more. I return to this issue in the next part of the dissertation.

The mathematics courses in Flickskolan

To give a general description of the mathematics courses in the girls' schools is not as easy as describing the mathematics courses in Folkskolan and Realskolan. The basic problem is that Flickskolan got its first national curriculum in 1928. Before that, there were no national directives regarding goals, teaching methods and courses.

Nevertheless, the mathematics courses at Flickskolan in 1928 were supposed to be a continuation of the courses in Folkskolan. The students could enter Flickskolan after the fourth year in Folkskolan, and they entered a 5-year practical program or a 7-year theoretical program. In the theoretical program the courses covered more topics than Folkskolan; during the third and fourth year, the students should study first-degree equations; during the sixth year, they should also study systems of first-degree equations. Moreover, during the sixth and seventh year they where introduced to theorems and constructions in geometry.[169] I suppose that these courses were based on the axiomatic method, but this is not stated in the curriculum. However, it appears as if the teachers at Flickskolan used the geometry textbooks intended for Realskolan. The reason for this is that there were, at least to my knowledge, no particular geometry textbooks for Flickskolan. The two textbooks intended for Flickskolan that I have found were also intended for Realskolan, which implies great similarities between the courses.[170] Yet, the geometry courses at Flickskolan cannot have been as extensive as the courses in Realskolan, since mathematics was given less time during the last two years at Flickskolan. (See the tables below.)

Time plans, 1905-1962

The tables below describe the number of mathematics lessons prescribed by the curricula during the period 1905 – 1965. Each lesson comprised 45 minutes and the numbers within the parentheses indicate the total number of lessons per week. In some cases, the total number of lessons could be larger if the students chose to study an additional subject. In these tables, the standard numbers are displayed.

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[169] SFS 1928:426, p. 1477; SFS 1950:61, p. 90
[170] One book was written by Anna Rönström and published in 1894. It was reprinted only in 1909. The other text book was written by Olof Josephson. Between 1900 and 1953, it was reprinted 15 times.

Table 2.

Folkskolan

grade	1	2	3	4	5	6	7	8	9
	kl 1	kl 2	kl 3	kl 4	kl 5	kl 6	kl 7	kl 8	kl 9
1919	3(24)	4(26)	4(28)	5(30)	5(30)	5(30)	5(30)		
1955	4(24)	4(26)	5(32)	5(34)	5(36)	5(36)	3(36)	3(36)	3(36)

Realskolan

grade	4	5	6	7	8	9
	kl 1(6)	kl 2(6)	kl 3(6)	kl 4(6)	kl 5(6)	kl 6(6)
1905	4(27)	5(29)	5(30)	5(30)	4(30)	5(30)

grade	5	6	7	8	9
	kl 1(5)	kl 2(5)	kl 3(5)	kl 4(5)	kl 5(5)
1927	5(37)	5,5(37,5)	4(39)	5(39,5)	5(37)
1933	4(36)	5(38)	4(38)	3(36)	4(34)
1955	4	5	3	4	4

grade	7	8	9	10
	kl 1(4)	kl 2(4)	kl 3(4)	kl 3(4)
1927	5,5(37,5)	4(39)	4(39,5)	5(38)
1933	5(38)	4(39)	3(37)	4(34)
1955	4	3	4	4

Flickskolan

	grade	5	6	7	8	9	10	11
		kl 1	kl 2	kl 3	kl 4	kl 5	kl 6	kl 7
theoretical program	1927	4(33,5)	5(34,5)	3(33,5)	4(35)	4(34,5)	3(~35)	3(~35)
practical program	1927	4(33,5)	5(34,5)	3(33,5)	4(35)	4(34,5)		
theoretical program	1950	4(33)	5(33)	3(34,5)	3(35)	3(34,5)	3(~35)	3(~35)
practical program	1950	4(33)	5(33)	3(34,5)	3(35)	3(34,5)		

Grundskolan

grade	1	2	3	4	5	6	7	8	9
	åk 1	åk 2	åk 3	åk 4	åk 5	åk 6	åk 7	åk 8	åk 9
1962	4(20)	4(24)	5(30)	5(34)	5(35)	5(35)	4(35)	4(35)	5[3](35)
1969	4(20)	4(24)	4(30)	5(34)	5(35)	5(35)	4(35)	4(35)	4(35)

The bracketed 3 in åk (=grade) 9, Lgr 62, denote the number of mathematics lessons at the practical programs. The 5 indicates the number of mathematics lessons at the program that prepared the students for further studies at the Gymnasiet.

If we consider the table for Folkskolan, the conditions for geometry instruction were changed due to the reform in 1955. Folkskolan was extended by one, and later on two, years between 1933 and 1955. As the total number of mathematics lessons was increased, new topics in basic algebra and geometry were added to the courses.

In Realskolan, the new time plan of 1933 brought considerable changes for geometry instruction. If we consider the five last years of Realskolan, i.e. grades 5 to 9, we can observe drastic changes of the number of mathematics lessons per week. The changes become even more drastic when we consider the last three years – it was during these three years, the axiomatic-deductive method was to be taught. The time for mathematics lessons changed in the following way with each reform.

Table 3.

reform	last 5	last 3
1927	2%	0%
1933	-18%	-21%
1955	0%	0%
1962	13%	15%
1969	-5%	-8%

The new curriculum of 1933 caused major protests, not just among the mathematics teachers in Realskolan. The number of lessons in mathematics and science subjects was reduced in both Realskolan and Gymnasiet. Because of this reform, a great number of teachers formed a national society for education in science and mathematics – *Föreningen för undervisning i matematik och naturvetenskap* – and its explicit aim was to create a public opinion for the restoration of mathematics and science as school subjects.[171] Actually, it appears as if their efforts were successful; in 1936, an extended mathematics course in Gymnasiet was created. In Realskolan, however, the time plan was not changed. Nor was a special course in mathematics introduced in Realskolan.[172]

The curricula and time plans did not stipulate how much time should be spent on arithmetic, algebra, geometry and various applications. An investigation done by Dahllöf (1960) during the late 1950's gives us an indication about the proportions. In Folkskolan, the teachers spent 15-20 percent of the lessons in mathematics on geometry. In Realskolan, the proportions were the following:[173]

Table 4.

Year in Realskolan	Proportions
3^d last year	20-25%
2nd last year	~35%
Last year	45-50%

According to Dahllöf (1960), the applied geometry exercises became more common than the theoretical ones during the last year in Realskolan.[174]

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[171] Lindholm (1991)
[172] Grimlund & Wallin (1939), pp. 322-324
[173] Dahllöf (1960), p. 441. The investigation of the distribution of arithmetic, algebra, geometry and applications is based on questionnaires from 1,597 teachers: Folkskola (228), Realskola (303), Praktisk Realskola (167), Flickskola (217), Enhetsskola (505), The schools in Stockholm in 1947 (177). [Dahllöf (1960), p. 67]
[174] Dahllöf (1960), p. 441

The debate on geometry instruction in Folkskolan

According to the yearbooks of the teacher education institutes for Folkskolan, the most commonly used literature on mathematics instruction was *Arbetssättet i folkskolan* (*Working methods of Folkskolan*); a treatise that also included essays on mathematics instruction.[175] Among these essays, there was one on geometry instruction.[176] Along with *Arbetssättet i folkskolan*, two treatises that focused on mathematics instruction in particular were used at the institutes: they were *Matematikundervisningen i folkskolan: metodiska råd* (*Mathematics teaching in Folkskolan: methodological advice*) by Gunnar Setterberg, first published in 1916, and *Den grundläggande matematikundervisningen* (*Elementary mathematics education*) by Frits Wigforss, first published in 1925.[177] Geometry instruction was also treated in these works.

The works of Setterberg and Wigforss were, however, not an instant success. During the 1920's, 1930's and early 1940's only two out of twelve institutes used their books. Not surprisingly, Setterberg and Wigforss worked as lecturers at these two institutes. However, during the 1940's, Wigforss' treatise became more popular and was included in teachers' training at other institutes as well. In the 1940's and 1950's five more editions of Wigforss' treatise were published.[178]

All the treatises mentioned above shared some similarities regarding goals and methodological issues. The authors base their accounts on a critique of rote learning. The common opinion was that learning by rote is counterproductive; the students just attain knowledge about facts, but not an ability to use these facts in different situations. Moreover, the authors believed that the students considered learning by rote as meaningless. In order to avoid this kind of learning, the teachers should arouse the students' interest for mathematics and make the students attain a more profound understanding of mathematics. In order to achieve this goal, the teaching methods should be founded on spatial intuition and self-activity. An important aspect is that a teaching method based on spatial intuition and visualisations was not just a matter of showing illustrations every now and then. On the contrary, illustrations, teaching material and experimental exercises were to be inserted at specific moments during a teaching sequence.

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[175] See Folkskoleseminariernas årsböcker. *Arbetssättet i folkskolan* contains one chapter for each school subject and each chapter comprises various essays on teaching methods. The essays on mathematics instructions was written by L. G. Sjöholm, N. E. Persson, Nils Alsén, Carl Gustaf Hellsten, Elsa Eriksson, Sven Wikberg, C. N. Hedegård, David H. Nissar, C. Lundahl and Edv. Steiner.

[176] This essay was written by C. Lundahl and Edv. Steiner.

[177] Setterberg also published a treatise named *Åskådlig matematikundervisning* (*Concrete mathematics teaching* or *Mathematics teaching by spatial intuition*), Setterberg (1913).

[178] Not till Folkskoleseminariernas årsböcker

In the following sections, I give a more detailed description of Wigforss' writings on these matters. In comparison, Setterberg and the authors of *Arbetssättet i Folkskolan* very much shared Wigforss' enthusiasm for ideas on mathematics instruction and spatial intuition. Moreover, their ideas on how teaching ought to be arranged were quite similar. The unique aspect of Wigforss was that he explained these ideas in more detail. He also submitted goals of mathematics instruction that was not mentioned in the curricula. Therefore, I have chosen to focus on the arguments of Wigforss in this thesis.

Wigforss on mathematics instruction

In comparison to the 1919 curriculum and the writings on the purpose of mathematics instruction, Wigforss gave expression to a somewhat different view. Along with the promotion of knowledge and skills suitable for daily life matters, Wigforss considered mathematics teaching to be a good opportunity to influence the students' intellects and spirits. This new purpose was saliently stated in the very first paragraph of the commentaries:

> § 1. The aim of mathematics teaching as well of the other school subjects may be said to be the communication of certain valuable knowledge to the children, but also to influence their mental faculties in a good way. … The possibilities of mathematics education to influence the intellects and spirits of the students must be considered significant. Hardly any of the other school subjects are more suitable to require the same clarity and orderliness of thought. Therefore, it must be considered as an essential task of mathematics teaching to educate the students with respect to logic. [179]

We find no formulations like this in the curriculum of 1919 or in the curriculum of 1955. However, in the preliminary curricula that preceded the coming Grundskolan, we find similar formulations.[180] Such preliminary curricula were produced in the first years of the 1950's and by the mid 1950's. They were also tested in numerous schools, the so-called *Enhetsskolor* throughout this decade. It is by no means surprising that these new goals appear in these curricula; Wigforss was very much involved in authoring them.

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[179] Wigforss (1925), p. 5: "§ 1. Målet för undervisningen i matematik såväl som i skolans andra ämnen kan ju sägas vara dels att bibringa barnen vissa värdefulla kunskaper och dels att påverka deras själskrafter i god riktning. ... Matematikundervisningens möjligheter att påverka elevernas tanke- och viljeliv måste anses betydande. Knappast något av skolans andra ämnen torde så bra kunna erfordra tankens reda och klarhet. Det måste därför betraktas som en väsentlig uppgift för matematikundervisningen att verka bildande på eleverna i logiskt avseende." We find a similar formulation in the fourth edition printed in 1952. [Wigforss (1952, pp. 5-6]

[180] Wigforss & Roman (1951) p. 5; Wigforss (1952), p. 5; Kungl. Skolöverstyrelsen (1956), p. 77. Noteworthy is that the formulation about training in logic is dropped in Kungl. Skolöverstyrelsen (1956).

In order to achieve these goals, mathematics instruction had to be based on spatial intuition and student activity; these were the basic components of Wigforss' methodology.

Intellectual training and visualizability

The argumentation regarding spatial intuition and shaping of the intellect was built around a critique of rote learning.

> The drumming of abstract doctrines is – if not preceded by clear observation – useless. [181]

> The children should not learn the concepts of surface, line, point by some abstract definitions, ...[182]

Still, Wigforss did not disapprove of mechanical reiterations, but if these are not preceded by a thorough understanding of concepts and techniques, they are thoughtless and therefore useless. Moreover, he considered mere mechanical repetition of rules in arithmetic training as a "meaningless game with numbers".[183] It was in connection to the efforts of developing the students' understanding of mathematics that Wigforss tied spatial intuition to intellectual training.

To Wigforss, proper intellectual training required that the students understand the content of the courses.

> Since the training of thought is a main purpose of mathematics teaching, the understanding of the content must be energetically aspired to, ...[184]

This understanding was achieved if the teaching was made concrete and possible to observe. *Åskådlig* was the Swedish term used by Wigforss. It may be translated by observable or visualizable.

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[181] Wigforss (1925), p. 114: "Ett inpluggande av abstrakta lärosatser är – om ingen klar åskådning ligger bakom – värdelöst."

[182] Wigforss (1925), p. 115: "Begreppen yta, linje, punkt skola barnen ej lära in genom några abstrakta definitioner, ..."

[183] Wigforss (1925), p. 6: " Visserligen är mekanisk säkerhet i denna teknik [räkneoperationer] nödvändig, men denna mekanisering kan och bör – åtminstone i väsentlig mån – komma som ett resultat av en ofta företagen upprepning av den tankegång, som ligger bakom tekniken. Alltså först begripande, sedan mekanisk färdighet så småningom. ... En stor vinning, som arbetet att förstå sättet för räkneoperationernas utförande för med sig, är att innebörden i dessa operationer därigenom belyses och blir förstådd av barnen, medan blott det mekaniska utförandet ur denna synpunkt är värdelöst, ja, skadligt, då tanklösheten lätt kan bli så stor, att såväl elever som lärare i det myckna sifferräknandet glömma att tänka efter, om själva *innebörden i de räkneoperationer*, som mekaniskt utföras, är begripen. ... Med ett sådant undervisningssätt börjar man betänkligt närma sig den gräns, där räkningen från att vara ett betydelsefullt arbete övergår till att bliva en meningslös lek med siffror." The italics are Wigforss's.

[184] Wigforss (1925), p. 5: "Då tankens skolning är en huvuduppgift för undervisningen i matematik, följer därur, att begripandet av kunskapsstoffet energiskt måste eftersträvas, ..." Wigforss uses the word "kunskapsstoffet", which I understand as *content* as I have defined it.

In order to make the students understand and profit from the course, *the teaching must be visualizable* [~åskådlig].[185]

By visualizing, Wigforss meant:

Visualizing is not just seeing, but having a clear, concrete conception, and the most visual [~åskådliga] conception is reached when the *children are active*, when they carry out the actions that visualize [åskådliggör] the operations of calculations themselves, instead of just remaining in their seats and watching the teacher carry out the actions.

The outer obvservation is however a means to reach the inner, to attain a visualization in our imagination, to attain a visual [åskådlig] way of thinking.[186]

If we follow Wigforss' argumentation in the first part of the quote, visualizing is not the same thing as simply looking at a picture; visualizing means that you perceive something particular in an illustration. Moreover, having a visualizable or concrete conception of something is the same thing as a having a clear conception, according to Wigforss.

If we recall Wigforss' writing on clarity and orderliness of thought in connection to the goals of mathematics instruction, this is what he meant by clarity. Hence, clarity is not primarily linked to logic or an idea about using precise language.

In the last part of the quote, we observe that there are two types of visualizing: outer and inner. My interpretation is that this is two modes of thinking, where the outer mode is a preparation for the inner mode. For instance, the illustration of the essential aspects of a proposition about something facilitates a more profound understanding of this proposition. Eventually, the illustration is not needed in order to understand this proposition, which means that the student has reached the inner mode of thinking.

Even though we may argue about the meaning of this quote, I think that it shows how Wigforss saw the aim of intellectual training as closely linked to his ideas on spatial intuition. Moreover, in comparison to the formulations in the curriculum, he gave the notion of spatial intuition in connection to mathematics instruction a more profound meaning. Mathematics instruction in Folkskolan should not only provide skills in calculations that were useful in every day life; it should also constitute a form of intellectual training.

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[185] Wigforss (1925), p. 8: "För att barnen skola kunna förstå och tillgodogöra sig kursen, *måste undervisningen vara åskådlig*." The italic is made by Wigforss.

[186] Wigforss (1925), p. 8: "Men åskåda betyder ej blott se, utan klart, konkret uppfatta, och den allra bästa åskådliga uppfattningen få barnen, när *de själva få vara verksamma*, när de själva få utföra de handlingar, som skola åskådliggöra räkneoperationerna, istället för att bara sitta och se på, när läraren utför dem. Den yttre åskådningen är emellertid ett medel att komma fram till den inre, till en åskådning i fantasien, till det åskådliga tänkandet." The italics are Wigforss's.

Visualizability and self-activity

As I mentioned above, Wigforss as well as Setterberg and the authors of *Arbetssättet i folkskolan* required that effective learning should be based on spatial intuition together with self-activity. To Wigforss, spatial intuition and self-activity were very much linked together. If we return to the quote above, Wigforss pointed out that the students' activity is crucial. Observing an illustration and perceiving something is an act where the student is intellectually active; she must think and draw conclusions on her own. Wigforss made the following distinction between passive and active thinking.

> The difference between a more active and a more passive mode of thinking is as noticeable as between observing and doing. Both modes are valuable, but in connection to school work there is seldom much room for active thinking. Therefore, studies in mathematics should, to some extent, constitute a counterweight in this respect, since the teacher can easily arrange the teaching in such a way that passive reception does not become the main feature.[187]

Another advantage of teaching based on self-activity was that it captured the students' interest. Wigforss makes the following statement regarding successful teaching and students' interests.

> The method that takes up the activity of the children will also become the most interesting, which is a thing of the greatest importance. Teaching that does not successfully capture the children's interest must be considered a failure in its essential parts.[188]

This was exactly what a teaching method based on spatial intuition provided.

An important aspect of Wigforss' view of spatial intuition and self-activity as teaching principles is that it contained an order for how teaching ought to be arranged: in order to explain a concept, a proposition or a method properly, you must begin with an illustration and the students' active observations. In this way, the students reach a clear and intuitive understanding of it. At best, the students will discover the novelty by themselves. To let the students make discoveries is what Wigforss calls a "heuristic method". Moreover, not until the student had gained an intuitive conception should he or she pass on to routine exercises.[189]

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[187] Wigforss (1925), p. 8-9: "Skillnaden mellan ett mera aktivt och mera passivt tänkande är lika markerad som mellan bara se på och själv göra. Båda formerna äro värdefulla, men i skolarbetet får ofta det aktiva tänkandet för litet utrymme. Matematikstudierna borde härvidlag i viss mån bilda en motvikt, då läraren lätt nog kan lägga undervisningen så, att det passiva mottagandet ej blir huvudsaken."

[188] Wigforss (1925), p. 9: "Den metod, som tar barnens aktivitet mest i anspråk, blir ock den intressantaste, en sak av allra största betydelse. Den undervisning, som ej lyckas fånga barnens intresse, måste anses vara i väsentlig mån misslyckad."

[189] Wigforss (1925), p. 9

By a comparison to the textbooks in Folkskolan during the period 1905-1962, we can observe that this routine was applied very often. I return to this issue in the next part of this thesis.

Wigforss on geometry instruction

According to the curriculum of 1919, the aim of geometry teaching was to convey basic knowledge about constructions, descriptions and calculations of geometrical magnitudes.[190] Next to this aim, there was a clear directive that the students should be prepared to apply this knowledge in daily life. In contrast, Wigforss added some other aims for geometry instruction that were not related to daily life.[191]

1. Geometry instruction in Folkskolan should convey certain basic knowledge that constitutes a necessary fundament for later education in several subjects.
2. Geometry instruction was supposed to train the students' space perception. Unfortunately, Wigforss did not make any further comments on the expression "conception of space".

Wigforss didnot, however, downgrade the value of promoting skills that are useful in daily life.

Regarding methodological issues, Wigforss mentioned only one leading principle apart from basic practical advice about not using too many complicated words and formulas, and that was visualizability, i.e. åskådlighet.[192] Yet, Wigforss did not give a more thorough account on the concept of visualizability in connection to geometry teaching; there were no extra general directives regarding how teachers should arrange their geometry lessons. He just states that the teaching should rest on spatial intuition and provides an illuminating example.

> However, a good result is, as always, dependent on how the teaching is carried out. The drumming of abstract doctrines is – if not preceded by clear observation – useless. What is the value of a child knowing that the size of a triangle is found by the multiplication of the base and half the height, if they at the same time are completely baffled when they face the task of finding the size of a triangular surface given in reality?[193]

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[190] Upf 19, p. 58: "Undervisningen i räkning och geometri i folkskolan har till uppgift att bibringa barnen en efter deras ålder och utveckling avpassad insikt och färdighet i räkning med särskild hänsyn till vad som erfordras i det dagliga livet ävensom någon förtrogenhet med geometriska storheters uppritning, beskrivning och beräkning."
[191] Wigforss (1925), p. 114
[192] Wigforss (1925), pp. 114-5
[193] Wigforss (1925), p. 114: "Men ett gott resultat är naturligtvis som alltid beroende av hur undervisningen bedrives. Ett inpluggande av abstrakta lärosatser är – om ingen klar åskådning ligger bakom – värdelöst. Vad värde kan det ligga i att barnen veta, att en triangels storlek finnes genom multiplikation av basen med halva höjden, och om de kunna uträkna sådana exempel i läroboken, där basens och höjdens längder finnas angivna, om de samtidigt stå

In this quote, Wigforss also suggests a particular routine for teaching and learning: before the students are introduced to area and volume formulas and exercises, they should have gained an ability to investigate geometrical objects on their own. We can also observe this routine in Wigforss' directives regarding how the lessons ought to be carried out. A majority of the lengthier directives includes a routine where the students first work through a set of experimental exercises; after that they are introduced to formulas and computational exercises. Take for instance the computation of the area of a cylinder.

> A suitable exercise is to draw the surface net of a cylinder and accomplish such a thing [a cylinder]. Through this it becomes clear how the surface of the cylinder is computed.[194]

Moreover, these directives were often accompanied by references to spatial intuition and ideas such as clear conception, self-activity, observation, and discovery.[195]

In comparison to the geometry textbooks of the period 1905-1962, the methodological suggestions made by Wigforss, but also the other authors of teaching literature on mathematics, appear to have been influential. The routine regarding the introduction concepts and formulas appears in more or less every textbook in some form. Nevertheless, we can hardly link the origin of this routine to Wigforss; textbooks that were designed according to this routine appeared well before Wigforss work on mathematics instruction. Therefore, I think it is more appropriate to consider Wigforss a person who summed up what the authors of geometry textbooks were already doing. His unique contribution was rather that he added a more theoretical flavour by linking this type of textbook design to ideas about visualizability and outer and inner visualizations.

The debate about geometry instruction in Realskolan

Axiomatic geometry and the training of reasoning

If we consider the curriculum for Realskolan, we find no formulations about geometry instruction and training in reasoning. In some of the contemporary geometry textbooks, though, the authors stated that geometry instruction,

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alldeles rådlösa inför uppgiften att taga reda på storleken av en i verkligheten given triangels yta?"
[194] Wigforss (1925), p. 125: " En lämplig arbetsuppgift är att rita ytnätet till en cylinder och förfärdiga en sådan. Härigenom blir det klart, hur cylinderns mantelyta skall beräknas."
[195] Wigforss (1925), pp. 115-127. I discuss Wigforss' particular directives in connection to the investigation of the geometry textbooks.

apart from providing knowledge in geometry, should train the students in logic and reasoning.[196]

In *Elementa* there was no real debate about geometry and the training of reasoning in the sense that someone disputed this idea. Even those debaters who had different opinions on other issues related to geometry instruction agreed about this idea. Those who treated the issue more in detail were Petrini, Hedström, and Olson.[197]

Olson's thinking on geometry instruction and training in reasoning contains two basic components. One is that geometry is the best subject for the training of formal reasoning. The other is that geometry instruction promotes a general intellectual ability to treat concepts and language in a formal manner, which leads to a general ability to perform sound reasoning. Olson summarizes his view on geometry instruction and training in reasoning in the following way:

> … the science of logic obtains its complete application on a relatively small content, and since, from the students' point of view, it treats easily demonstrable and tangible realities, there are always pleasing opportunities to display absurdities for the undeveloped mind, which may lead him to incorrect conclusions. The formation of concepts and the art of defining, judging, and deducing and making logical systems are cognitive operations, to which the students are accustomed through studies in geometry. The students become accustomed to using clear and distinct concepts; they get an awareness of the importance of using correct expressions: not to say too much or too little. Thus, these requirements [on geometry instruction] sustain the fostering of the students' feeling for language and their ability to use language in a correct manner.[198]

Here we should note that, in the first sentence, Olson stressed that geometry, due to its limited and concrete content, is the ultimate subject to train students in reasoning. In doing so he pointed out a unique feature that distinguished geometry from other school subjects.

Hedström and Petrini, on the other hand, were not as explicit as Olson about why geometry was the most suitable school subject for training rea-

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[196] See for instance Sjöstedt (1936), p. 8; Vinell (1907), p. 6; Olson (1940), p. V
[197] Actually, Petrini did not convey his idea about geometry and training in logic and reasoning in *Elementa*. This particular subject was treated in a periodical intended for teachers in Folkskolan, *Folkskollärarnas tidning*. *Folkskollärarnas tidning* (1921), is. 46, pp. 647-649.
[198] Olson (1926/27), p. 14: "Den logiska vetenskapen får här sin så gott som fullständiga tillämpning på ett relativt litet sakinnehåll, och emedan det rör sig om för lärjungarna lätt påvisbara, påtagliga realiteter, finns det ständigt bekvämt tillfälle att praktiskt påvisa för det logiskt outvecklade sinnet orimligheter, till vilka hans tankefel leda honom. Begreppsbildningen och definierandet, omdömesbildningen och slutledningen samt den logiska systembildningen äro tankeoperationer, med vilka lärjungarna genom geometristudiet bli förtrogna. Lärjungarna vänjas att röra sig med klara och tydliga begrepp, får känsla för betydelsen av att uttrycka sig korrekt: att varken säga för mycket eller för litet. Fordringarna härpå bidra således till att fostra lärjungarnas språkkänsla och deras förmåga av korrekt språkbehandling"

soning. Instead, they argued that geometry instruction promoted a critical mind in the sense that the students became aware of and confident about their own reasoning. Hedström wrote:

> His imagination and power of initiative ought to be trained. He should then, by making observations and drawing conclusions, reach a certain degree of independent opinions and conceptions. We should foster our students not only to be self-critical, but also to trust themselves.[199]

Petrini argued that mathematics education, just like grammar in classical languages, radicalized humans; it made them think in an efficient manner. As a contrast he mentioned the "spineless" studies in history and literature.

> It is the incoherent rote learning of the humanities and first and foremost the spineless studies in history and literature that turns the mind away from actual reality and its laws. Mathematics and science make thought accustomed to: this is how it m u s t be; the humanities and its lack of necessity often end up in: this is as good as the other.[200]

Even though Petrini wrote about reasoning and mathematics education in general, it is hard to believe that he should have excluded geometry in this discussion. In his articles in *Elementa*, he pointed out that Euclid's *Elements* is the ideal textbook since it is complete with respect to logic.[201]

> On the other hand, you cannot find one single logical error in Euclid; his system is founded on the most complete logical distinctness in every detail.[202]

Also Olson argued that geometry instruction should imbue stuents with a critical attitude towards reasoning and language.

> The tendency to generalize is one of many human vices and it has, as we know, caused much misery on our earth. In mathematics, one has many convenient opportunities to show how far afield this may lead us. If one asks a boy, in how many points two circles can dissect each other, he answers with-

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[199] Hedström (1919), pp. 196-197: "Hans fantasi- och initiativförmåga bör uppövas. Han skall genom att iakttaga och draga slutsatser skaffa sig en i någon mån självständig åsikt och uppfattning. Vi skola uppfostra våra elever ej endast till självkritik utan också till självtilllit."
[200] *Folkskollärarnas tidning* (1921), iss. 46, p. 648: "Det är den sammanhangslösa humanistiska minneskunskapen och främst den ryggradslösa historie- och litteraturläsningen, som vänder sinnet bort från den faktiska värkligeheten och dess lagar. Matematik och naturvetenskap vänja tanken vid ett: så m å s t e det vara, humaniora med dess frånvaro av nödvändighet mynna ofta ut i ett: det är lika bra det ena som det andra."
[201] Petrini (1918), p. 194
Petrini (1924/25), p. 134
[202] Petrini (1924/25), p. 130: "Däremot kan man hos Euklides själv inte upptäcka en enda logisk felaktighet, utan är hans system uppbyggt med den mest fulländade logiska skärpa på varenda punkt."; p. 134: "Såvitt jag kan se, finns det ingen möjlighet att rubba på det grundläggande Euklidiska systemet det ringaste grand utan att samtidigt pruta på strängheten."

out blinking: *two!* How do you know? Well, it is obvious; one sees it as one makes a drawing:
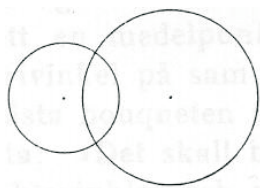
It is said that it is bad to sow the weed of doubt in the souls of innocent children, but, when it comes to altering their naïve beliefs in their outer senses, I think it is justified. Ask him then to draw two circles (preferably of equal size, to avoid the case of tangents) where the centers are rather close in comparison with the length of the radiuses; spatial intuition then shows that the circles have not only two points but whole arcs in common. This entail, at the very least, that the answer can be discussed. Clearly, such discussions will arouse the students' interest in having the case proven.[203]

An important aspect of the argumentation of Olson, Hedström, and Petrini is that they assigned a general usefulness to geometry instruction. The ability to reason effectively, to think critically, and to treat concepts and language carefully is indeed useful in other areas as well, not just mathematics. Actually, in comparison, explicit arguments where geometry instruction in Realskolan was supposed to provide valuable knowledge in mathematics were quite rare. Of course, that idea may have been taken for granted.

So, was this intellectual training a goal intended for the most theoretically gifted students aiming for the Gymnasium and later on the university? Or did this goal also include those students who entered vocational programs or working life after Realskolan? I would say that this goal included all the students in Realskolan, at least until the early 1950's. I have two arguments for this claim.

1. Until 1951, there was only one course program in mathematics; hence, there was no differentiation where the students could chose between difficult or less difficult courses. Even when practical programs were introduced in 1934, there was only one course program in mathematics.[204]

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[203] Olson (1926/27b), p. 78: "Att gärna vilja generalisera är nu en av människans många odygder och har som bekant ställt till mycket elände på vår jord. Inom matematiken har man många bekväma tillfällen att visa, hur galet dylikt kan leda. Om man frågar en pojke, i hur många punkter två cirklar kunna skära varandra, svarar han utan att blinka: *två!* Hur vet du det? Jo, det är ju självklart, det ser man ju bara man ritar: ... Det sägs, att det är av ondo att så tvivels ogräs i oskyldiga barnasjälar, men då det gäller att rubba deras kolartro på de yttre sinnena, tror jag att det kan vara berättigat. Bed honom då att rita två cirklar (helst lika stora, så att tangentfallet undvikes) med medelpunkterna rätt nära varandra i förhållande till radiernas längder; åskådningen visar ju då, att de båda cirklarna ej hava blott två punkter utan hela bågar gemensamma. Och i och med det är ju frågan i alla händelser diskutabel. Det är ju klart, att genom sådana diskussioner väckes lärjungarnas intresse för att få saken bevisad."

[204] See the chapter above on the curricula for Realskolan.

2. Between 1917 and 1962, the final exams in mathematics included geometry exercises where the students were supposed to motivate some propositions. Until 1950, more than 80 percent of the students that entered the exam process took the final exams in mathematics.[205]

Petrini even argued that the axiomatic approach made the teaching easier for the less gifted students, since it brought clarity and order.[206]

Another aspect of the debates in *Elementa* on geometry instruction is the type of arguments the debaters did not use. Here we can note that arguments about the preservation of some kind of cultural heritage were not used in the debate. Even a man like Petrini, who vigorously defended Euclid's *Elements* and considered it the most ideal textbook, did not use this type of argumentation. His basic argument was that geometry instruction should train the individual student in reasoning.

If we consider the debate during the 1950's that preceded the new curriculum for Grundskolan and the cancellation of the axiomatic approach to school geometry, the idea about training in rational reasoning was the main argument of those who wanted to keep the axiomatic method in some way or another.[207]

The debate about textbooks and teaching methods, Part I

The debate in *Elementa* on geometry instruction comprises two episodes, 1917-1927 and 1938-1939. During the first episode, there are eight articles that are linked to each other; during the second episode, there are seven. Apart from these, there are some minor articles on geometry instruction, but they are not directly linked to the other articles.

The first episode was initiated by Petrini as he defended Euclid's *Elements* as the ideal textbook in two articles, criticizing contemporary textbooks and teaching methods. His basic criticism was that textbooks that deviated from the disposition of the *Elements* were less rigorous. Moreover, Petrini argued that no pedagogical benefits were achieved by lowering the level of rigor. Petrini got reactions from Meyer, Hedström, and Olson; all three disputed that Euclid's *Elements* was the ideal textbook for elementary instruction. Moreover, they claimed the students were not able to grasp the rigor Petrini was aiming at.

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[205] See the chapter below on the final exams in mathematics.
[206] Petrini (1918), p. 203
[207] Ullemar (1957), p. 4; Sjöstedt (1961), p. 115. These articles did not appear in *Elementa*. Actually, no longer articles related to the debate on geometry instruction appeared in *Elementa* during the 1950's. Ullemar submitted his article in the periodical *Tidskrift för skolmatematik* (Journal of School Mathematics) and Sjöstedt in *Pedagogisk debatt* (Pedagogical Debate). In these journals other articles on geometry instruction were submitted as well.

Petrini's defense of Euclid's Elements

Petrini considered Euclid's *Elements* to be the ideal geometry textbook from a logical point of view.

> … you cannot find one single logical error in Euclid; his system is on each point founded on the most complete logical distinctness. [208]

This standpoint is a bit surprising. A person with a PhD in mathematics and an interest in geometry cannot have been unaware of the criticisms about Euclid's tacit assumptions and the following logical inconsistencies. Petrini even devoted a whole article in *Elements* to a discussion on modern geometry and Hilbert's *Grundlagen der Geometrie*.[209] Moreover, on no occasion did he dispute Hilbert's or anyone else's criticisms of Euclid. Therefore, when he writes that "you cannot find one single logical error in Euclid", he may have meant that each explicit justification of a statement is done in a logically correct manner.

Regarding the works of Hilbert, Petrini claimed that the *Grundlagen…*could be considered an endpoint of research on the fundaments of geometry.[210] Moreover, in Petrini's view, Hilbert's axioms were purely mathematical and independent of experiences of physical objects; in this way, Hilbert had shown that it was possible to think mathematically about geometry without any experience of physical objects.

> [The axioms are] completely independent of empirically found properties of physical bodies. Hence, he has shown that a geometry is mathematically possible without any experience of solid bodies.[211]

Owing to its abstract character, Petrini found Hilbert's axiomatic system to be inappropriate for elementary instruction in Realskolan.

> Due to its abstract formulations, the axiomatic system of Hilbert seems to be less appropriate as a foundation of geometry for beginners, even for the fourth class in Realskolan some of his axioms appear altogether unnecessary to the beginner, others, such as III:6, too far-fetched. Here, as in all other cases, one ought to build on what is known and one ought to make the first

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[208] Petrini (1924/25), p. 130: "Däremot kan man hos Euklides själv inte upptäcka en enda logisk felaktighet, utan är hans system uppbyggt med den mest fulländade logiska skärpa på varenda punkt."

[209] Petrini (1917), pp. 197-207

[210] Petrini (1917), p. 197

[211] Petrini (1917), pp. 205-206: "[Axiomen är] alldeles oberoende av rent empiriskt funna egenskaper hos de fysiska kropparna. Härmed har han i sak uppvisat, att en geometri är matematiskt tänkbar utan erfarenhet av fasta kroppar."

basic instruction *as visual* [~åskådlig] *as possible without compromising exactness.*[212]

In Petrini's view, the value of Hilbert's work had more of an ideological character: teachers and textbook authors should be inspired by its rigor and try to make Euclid's account even more rigorous.

> On the contrary, it is the present generation's obligation to make the Euclidean account, if possible, even more exact. This is nowadays feasible due to the scientific investigations in the area of geometrical axioms of the last century. These investigations have been brought to a certain completion by Hilbert's award-winning work >>Grundlagen der Geometrie>>. In this work, the guidelines are also drawn for all future accounts on the elements of Geometry that aspire to be exact.[213]

Still, it was not possible to modify the outline of Euclid's *Elements*.

> As far as I can see, there is no possibility of changing the fundamental Euclidean system the slightest without reducing the rigorousness at the same time.[214]

Regarding the contemporary textbooks, Petrini identified four types of flaws that were linked to alterations in the Euclidean system.[215]

1. If a theorem is proved by means of the fifth postulate, i.e. the parallel postulate, the theorem becomes less general than if the fifth postulate is not used. If the fifth postulate is not used, the theorem applies for Euclidean geometry as well as non-Euclidean geometries. Thus, proofs where the fifth postulate is included becomes less general. Moreover, the teachers lose a good opportunity to discuss theorems that apply to non-Euclidean geometries.

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[212] Petrini (1917), p. 206: "På grund av sin abstrakta formulering synes det Hilbertska axiomsystemet för mången vara föga ägnat att bygga geometrin på för nybörjare eller ens i realskolans fjärde klass, helst somliga av hans axiom förefalla nybörjaren alldeles onödiga, andra såsom III:6 alltför långsökta. Här som alltid eljest bör man bygga på vad som är känt och göra den första grundläggande undervisningen så *åskådlig som möjligt utan att inskränka på exaktheten.*" The italics are done by Petrini.

[213] Petrini (1918), p. 195: "Tvärtom är det den nuvarande generationens skyldighet att om möjligt göra den Euklideiska framställningen ännu mer exakt. Detta är numera utförbart på grund av det sista århundradets vetenskapliga forskningar på de geometriska axiomens område. Dessa undersökningar ha bragts till en viss avslutning genom Hilberts prisbelönade arbete >>Grundlagen der Geometrie>>. Med detta arbete äro också riktlinjerna dragna för alla framtida framställningar av geometrins element med anspråk på exakthet."

[214] Petrini (1924/25), p. 134: "Såvitt jag kan se, finns det ingen möjlighet att rubba på det grundläggande Euklidiska systemet det ringaste grand utan att samtidigt pruta på strängheten."

[215] Petrini (1924/25), pp. 131-133

2. Theorems proved by Euclid are presented as axioms, as for instance the proposition that the sum of two sides in a triangle is greater than the third. According to Petrini, as few axioms as possible should be used.
3. The construction theorems serve as proofs of existence in the *Elements*. This makes it almost impossible to alter the order of the theorems.
4. Translations or movements of figures in the plane should not be used. Here, Petrini suggested that theorem I.4 should be considered an axiom.[216]

We can compare this list to how Asperén and Olson designed their textbooks, which is discussed in Part D of this dissertation. Their works seem to be the subject of Petrini's criticisms. At the time of Petrini's articles, Asperén's geometry textbook was the most popular one in Sweden.

Nevertheless, Petrini's defence of Euclid *Elements* as a textbook was not only based on purely logical considerations. He also saw pedagogical advantages in a course that followed the *Elements* – the rigorous approach was pedagogical in it self.

As I have mentioned earlier on, the students in Realskolan began their studies in geometry with a course in measuring and computing lengths, areas, and volumes. During this course, propositions were established by empirical investigations, and there was no axiomatic approach. This course also included practical applications. Like all the other debaters, Petrini found this course necessary since it was an opportunity for the students to get acquainted with the basic geometrical concepts. A crucial issue in the debate in *Elementa* was how the introduction of the axiomatic method ought to be carried out. Petrini's idea was that the students should be introduced to the axiomatic method without any further ado.

> With beginning in the fourth grade and without any scruples, [one should] introduce a scientifically exact teaching method. On the other hand, any underestimation of the disciples' intellectual abilities will lead to wobbly drivel, which will have an impeding effect on their spiritual development towards standing on their own feet. One should therefore reject all attempts to make geometry easier to understand by means of bargaining away the rigor of Euclid's account.[217]

The less gifted students in particular would benefit from such an arrangement. If the basic concepts and methods were examined in a vague fashion,

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[216] Most likely, Petrini was here referring to Hilbert's axiom III:6: If so, the suggestion is a bit surprising since Petrini, seven years earlier, considered this axiom too far fetched for the students at Realskolan.

[217] Petrini (1918), p. 195; [Man bör] utan skrupler slå in på en vetenskaplig exakt undervisningsmetod från och med den fjärde klassen. Underskattandet av lärjungarnas förståndsmognad leder däremot till ett löst pjoller, som endast verkar hindrande på deras andliga utveckling till att stå på egna ben. Man måste sålunda förkasta alla försök att göra geometrin lättfattligare genom att pruta av på strängheten i Euklides' framställning."

these students would get stuck at an early stage; the more gifted students, on the other hand, would cope with the vague explanations.[218]

What Petrini meant by "bargain away the rigor" is probably related to his criticisms of the contemporary textbooks mentioned above. The meaning of "wobbly drivel" is however not obvious by his writings. However, the articles by his adversaries Meyer, Hedström, and Olson provide a good picture of the target for Petrini's criticisms.

The alternative approach of Meyer, Hedström, and Olson

In the articles by Meyer, Hedström, and Olson, two issues were treated in particular. 1) The transmission from a less formalistic geometry course to a much more formalistic course, i.e. the introduction of the axiomatic method. 2) Euclid's *Elements* as a textbook at elementary level.

"Gentle" transmission as an introduction to axiomatic geometry

Their basic standpoint was that the transmission had to be done in a gentler manner or "continuously" and "mildly" as Hedström put it.[219] All three very much disputed Petrini's idea of entering a scientifically exact teaching method in what they considered an abrupt manner. Moreover, they considered Euclid's *Elements* to be inappropriate as an elementary textbook, and they refuted Petrini's idea that an exact scientific method would make geometry instruction more accessible. According to them, the teachers' task was not to introduce science, i.e. the axiomatic method, in its most complete form. Instead, the teachers ought to introduce the students to the value of a formalistic way of reasoning. In the following sections, their standpoints are described more in detail. Let us first consider the notion of so-called "gentle" transmission.

A key point in the argumentation of all three was that the students have to understand the importance of the axiomatic method – what is its purpose?[220] If not, a majority of the students would quickly lose interest in geometry and formal reasoning. How this ought to be accomplished was treated by Hedström and Olson. Their basic idea was that the students, even after the practical course in geometry, should be introduced to new theorems by means of empirical investigations, e.g. measurements or speculations based on spatial intuition. Hence, both Hedström and Olson underscored the importance of spatial intuition in connection to the courses in axiomatic geometry.[221] At best, the students should even discover the theorems on their own. In this way, the students would get a better understanding of the meaning of the theorems, than if the theorems were established via a proof without any fur-

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[218] Petrini (1918), p. 203
[219] Hedström (1919), p. 195
[220] Hedström (1919), pp. 195-196; Meyer (1924/25), p. 140; Olson (1926/27a), p. 23
[221] Hedström (1919), pp. 198-200; Olson (1926/27a), pp. 15-16

ther explanation. Then, as the teachers confronted the students with questions about the truths of the theorems, they should realize that the empirical methods are defective, e.g. inductive inferences are not perfectly reliable, observations and intuitive reasoning may lead wrong, measuring by means of a ruler or a protractor is not an exact method, etc.[222] Here are two examples that Olson mentioned, which also are good examples of how he believed geometry instruction imparts a critical attitude:

> For instance the theorem on vertical angles. After they have used the protractor to measure and they have found out that the angles are equal, one may draw the diagram on the board and ask the students if it is possible to realize the equality between the angles without measuring? An answer one then should expect is that it obvious even without measuring. This has to be refuted by a counterargument, if someone dares to say that it is obvious that they are *exactly* equal. Is it not thinkable that one of the angles is half a degree greater than the other, without our noticing it, unless one has measured carefully? *If* then someone should object that one cannot rely on careful measuring, since it is impossible to detect differences of about 1/10 °, then one has to be very satisfied, otherwise one has to guide the students to this reflection.[223]

> The tendency to generalize is one of many human vices and it has, as we know, caused much misery on our earth. In mathematics, one has many comfortable opportunities to show how wrong the like may lead. If one asks a boy, in how many points two circles can dissect each other, he answers without blinking: *two!* How do you know? Well, it is obvious; one sees it as one makes a drawing:
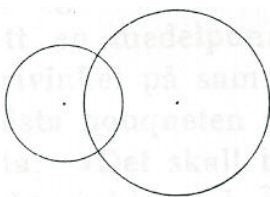
> It is said that it is bad to sow the weed of doubt in the souls of innocent children, but, when it comes to altering their naïve beliefs in their outer senses, I think it is justified. Ask him then to draw two circles (preferably of equal size, to avoid the case of tangents) where the centers are rather close in com-

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[222] Hedström (1919), pp. 199-200; Olson (1926/27a), pp. 77-78

[223] Olson (1926/27a), p. 23: "T. ex. satsen om vertikalvinklar. Sedan de genom mätningar med gradskivan funnit, att de äro lika stora, kan man rita figuren på tavlan och fråga lärjungarna, om man inte utan mätning skulle kunna ha insett att vertikalvinklarna äro lika stora? Ett svar, som man då har att vänta, är, att det syns utan vidare, utan mätningar. Detta får vederläggas med motfrågan, om man vågar säga att det syns, att de äro *precis* lika stora. Kan det ej tänkas att den ena är en halv grad större än den andra, utan att detta märks, såvida man ej gjort en noggrann mätning. *Om* då någon skulle komma på invändingen, att då man ej heller litar på mätningen, ty skillnader på t. ex. 1/10 ° kunna ej då upptäckas, får man vara mycket nöjd, annars får man själv leda lärjungarna till denna reflexion."

parison with the length of the radiuses; spatial intuition then shows that the circles have not only two points but whole arcs in common. This entail, at the very least, that the answer can be discussed. Clearly, such discussions will arouse the students' interest in having the case proven.[224]

In this way, the students were supposed to realize that the significance of formal proofs is the most certain method to establish the truth of a theorem. Moreover, they should also discover that it sometimes was a more convenient method in comparison with tedious and numerous measurements.

The idea of a less formal preparatory course and 'gentle' transmission was by no means a novelty presented by Meyer, Hedström, and Olson in the 1920's. For instance, in an official commission on textbooks in 1871, the investigators express a wish that geometry instruction should be organized in a similar way.[225]

Meyer's critique of Euclid's Elements as a textbook

The critique of Euclid's *Elements* as an elementary textbook was predominantly delivered by Meyer. However, Hedström's and Olson's ideas about "gentle" transmission are in line with this critique in the sense that all three underscored that a high level of rigor would not alone facilitate effective geometry instruction. Moreover, in his geometry textbook, Olson pointed out that he had picked up many ideas from Meyer's geometry textbook.[226]

In Meyer's argumentation there is a clear distinction between logical and pedagogical requirements. According to Meyer, Euclid had cared too much about logic and he claimed that the theorems were organized in a certain way because of how they were used in later proofs. This hampered the account from a mathematical point of view; theorems that treated the same concepts or objects were kept apart, for instance the theorems on congruency (I.4, I.8, I.26) or theorems on the relationship between a straight line and a circle (III.2, III.5-6, III.10-13, III.16-19). This disposition also made it hard for the students to get an overview of the subject and to understand geometry.[227]

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[224] Olson (1926/27b), p. 78: "Att gärna vilja generalisera är nu en av människans många odygder och har som bekant ställt till mycket elände på vår jord. Inom matematiken har man många bekväma tillfällen att visa, hur galet dylikt kan leda. Om man frågar en pojke, i hur många punkter två cirklar kunna skära varandra, svarar han utan att blinka: *två!* Hur vet du det? Jo, det är ju självklart, det ser man ju bara man ritar: ... Det sägs, att det är av ondo att så tvivels ogräs i oskyldiga barnasjälar, men då det gäller att rubba deras kolartro på de yttre sinnena, tror jag att det kan vara berättigat. Bed honom då att rita två cirklar (helst lika stora, så att tangentfallet undvikes) med medelpunkterna rätt nära varandra i förhållande till radiernas längder; åskådningen visar ju då, att de båda cirklarna ej hava blott två punkter utan hela bågar gemensamma. Och i och med det är ju frågan i alla händelser diskutabel. Det är ju klart, att genom sådana diskussioner väckes lärjungarnas intresse för att få saken bevisad."
[225] *Underdånigt Betänkande afgifvet den 9 Oktober 1871...*
[226] Olson (1940), pp. v-vi
[227] Meyer (1924/25), p. 142

Another line of criticism that Meyer followed concerned the complexity and awkwardness of some of Euclid's proofs. Meyer argued that the proofs should be more in line with the "real grounds" and the "nature" of the theorems. He even implied that a good proof should reveal some sort of causality.[228]

> A similar tendency to mess up the proofs is found also in I:5, where he [Euclid] unnecessarily extends the sides in order to get inverted congruent triangles that do not wholly, but only partially, cover each other. Thereby, the theorem will not be taken from its real grounds, that the triangle is inversely congruent to itself or, according to another proof, that it can be folded together, without taking it from superficial grounds that have little to do with the nature of the theorem.

> This is a common feature of Euclid, and it constitutes a major error from our point of view. On the question: "How come the angles at the base of an isosceles triangle are equal?" nobody can answer that it is because when the sides are expanded and the linkage is made crosswise, then two pairs of converted and congruent triangles occur, instead you then have to answer either that the diagram is symmetrical (= it can be folded together along a symmetry line) or that it can be convertibly laid on itself.[229]

Following Meyer's quote above, he suggested the following movement and construction.
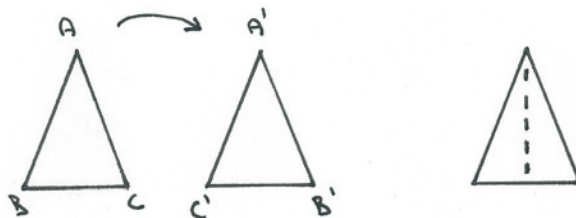
[228] This type of criticisms against the causality of the proofs in Euclid's *Elements* is by no means typical for the 20th century. Mancosu (1996) describes how mathematicians and philosophers in the 17th century debated whether a proof in classical Euclidean geometry is casual or not. Then, the underlying issue concerned the scientific status of geometry in relation to Aristotle's policies on science. [Mancosu (1996), pp. 12-24]

[229] Meyer (1924/25), pp. 141-142: " En snarlik tendens att krångla till bevisen finnes även i I:5, där han onödigtvis drager ut sidorna för att få de omvänt kongruenta trianglarna att ej helt, utan endast delvis, täcka varandra. Härigenom kommer satsen att ej tagas ur sina verkliga grunder, att triangeln är omvänt kongruent med sig själv eller, enligt ett annat bevis, att den kan sammanvikas, utan den tages fram ur uppkonstruerade grunder, som föga hava med satsens natur att skaffa. Detta är överhuvudtaget vanligt hos Euklides, och utgör i vår tanke ett stort fel hos honom. På frågan: >>Hur kommer det sig till, att vinklarna vid basen i en likbent triangel äro lika?>> kan nämligen ingen människa svara att det beror därpå, att om sidorna utdragas och sammanbindning sker korsvis, så uppkomma två par omvänt kongruenta trianglar, utan man måste svara, antingen att den är en symmetrisk figur (= kan sammanvikas utefter en symmetrilinje) eller att den kan läggas omvänd på sig själv."

In several of the contemporary textbooks, proofs of the theorem on isosceles triangles were indeed based on the movement or the construction suggested by Meyer. I return to these proofs in the next part of the thesis.

The proof of Euclid that Meyer was referring to included the following diagram where the equal sides *AB* and *AC* are equally extended and the lines *FC* and *GB* are drawn (You find a complete proof in Appendix A.)
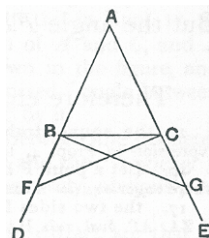
Meyer also criticized some of Euclid's definitions, for instance the definition of an angle.

> A *plane angle* is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.[230]

He did not like the concept *inclination* since it was synonymous with *angle*. Moreover, the definition gave no hint of how the inclination was measured.[231] Instead he suggested a definition based on the concept *rotation*:

> [An angle] is the rotation that one of the lines needs to undertake, in order to coincide with the other.[232]

Regarding Petrini's criticisms of contemporary textbooks, which probably included Meyer's own, Meyer countered them as well. Petrini argued that the structure of Euclid's *Elements* could not be altered without losing some of its rigor, and he explicitly mentioned what he saw as misuses in various textbooks; in particular, Petrini criticized the misuse of the parallel postulate, the construction theorems, and translations in the plane. Meyer's opinion on these issues was the following.

- From a pedagogical point of view, he rejected Petrini's objection that the initial theorems become less general if the parallel postulate was used earlier on; the question whether or not a theorem was valid in the Euclidean as well as non-Euclidean geometries was not

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[230] Heath (1956), vol. 1, p. 176
[231] Meyer (1924/25), p. 140
[232] Meyer (1924/25), p. 141: "[En vinkel är] den vridning, som den ena linjen behöver undergå för att sammanfalla med den andra."

relevant to students at the elementary level. Thus, the parallel postulate could be applied earlier on in comparison with Euclid.[233]

- Meyer applied a similar argument regarding the issue on construction theorems as proofs of existence, in this case theorem I.2-3 and I.9-12. He argued that the students did not understand arguments about existence. Moreover, some of the constructions theorems, theorem I.2-3, appeared especially awkward to the students since these kinds of constructions were easily done by means of a graded ruler in a practical context. Thus, theorems I.2-3 could be omitted and one did not have to consider the question of existence.[234]

- He did not approve of the idea that Euclid avoided translations of triangles and superposition due to the fact that he used it only in connection with the proofs of the theorems I.4 and I.8. Since the theorems are used in several proofs, translations of triangles and superposition are applied indirectly. Thus, translations of triangles and superposition could be used without hesitation if one allowed it in the proofs of theorems I.4 and I.8.[235]

If one considers contemporary textbooks, several authors seem to have shared Meyer's view, as they on several points did not follow Petrini's recommendations, among them Asperén, Olson, and Sjöstedt, who wrote some of the most popular geometry textbooks. I discuss their textbooks more in detail in Part D of this thesis. Actually, Meyer too authored a geometry textbook, but it did not gain any popularity.[236]

Olson on movements of geometrical objects and the question of putting pedagogical requirements before scientific

In his article on the goals and methods of geometry instruction, Olson emphasized the need to make a distinction between school geometry and geometry at a scientific level. The distinction was based on three issues.

1) The first issue concerned the use of movements. Here, Olson pointed to two scientific traditions: those mathematicians who did not use the concept at all in their geometries and those who did. As an example of the latter tradition, Olson mentioned Hilbert and his work "Grundlagen der Geometri". As an example of the former tradition he mentioned Helmholtz, as well as Peano. Olson's main point, though, was that the allowance of movements provided pedagogical advantages. In particular, since movements or motions appealed to spatial intuition.[237]

[233] Meyer (1924/25), pp. 143-144
[234] Meyer (1924/25), p. 140
[235] Meyer (1924/25), p. 141
[236] Meyer (1909)
[237] Olson (1926/27), pp. 74-75

And we may argue about the scientific aspects of the matter, but from a pedagogical point of view, one has to admit that it is to deprive geometry of one of the most primitive concepts of our spatial intuition, when one tries to build it up ignoring any kind of movement. Because it is through the different positions of items in space that our perspective seeing is trained. In order to get a geometrical conception of an item, we want, so to speak, to see it from all sides, which takes place via transportations of ourselves our the item. The procedure of generation, that a line is formed by the movement of a point or a surface by a line, is too fruitful to our geometrical conception and seems so natural, that any static geometry appears as something unnatural, something that has nothing to do with our mobile world. And since it should be impracticable to introduce a geometry of Hilbert with its 21 axioms, when it comes to the development of the disciples' minds, there is no reason to stop at half way with Euclid, but it would be best if we first, as well as last, let the concept of movement play the most significant role that it deserves from a practical as well as pedagogical point of view.[238]

Thus, Olson legitimized the use of movements in elementary textbooks by making references to scientific works of Helmholtz and Peano. Another argument was that geometries that did not include motions or movements of geometrical objects seemed unnatural, and this did not appeal to our spatial intuition.

2) The second issue concerned the introduction of definitions and axioms, especially the general axioms on magnitudes.[239] According to Olson, you should not introduce definitions and axioms together in the beginning of the course. They should rather be introduced when needed. Otherwise, the students might be confused by the abstract formulations, even though they might have an intuitive understanding of the axioms. Moreover, the students should in general not be encouraged to refer to the axioms when they carried out proofs. Instead the teacher would consider the confidence by which the student carried out a proof. In cases where a student was confused, the teacher might be forced to consider the axiom in question.[240]

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[238] Olson (1926/27), pp. 74-75: "Och man må tvista om sakens vetenskapliga sida, ur pedagogisk synpunkt måste man medge, att det är att beröva geometrien ett av åskådningens allra primitivaste begrepp, då man försöker bygga upp den bortseende från all rörelse. Det är ju genom föremålens olika plats i rummet som vårt perspektiviska [sic] seende utbildas. För att få en geometrisk uppfattning av ett föremål vilja vi så att säga se det från alla sidor, vilket sker genom förflyttning av oss själva eller föremålet. Generationsförfarandet, att en linje bildas genom en punkts rörelse eller en yta genom en linjes, är ju också för vår geometriska uppfattning alltför fruktbringande och tyckes oss så naturligt, att varje stel geometri ger oss intryck av något overkligt, som ej har med vår rörliga värld att skaffa. Och när det ändå, då det gäller våra lärjungars förståndsutveckling, torde det vara ogörligt att införa en Hilberts geometri med sina 21 axiom, finns det intet skäl att med Euklides stanna på halva vägen, utan det torde vara bäst att först som sist låta rörelsebegreppet i geometrin spela den väsentligaste roll, den ur praktisk och åskådlighetssynpunkt förtjänar att göra."
[239] Olson used the term "arithmetical axiom". In his textbook, though, the first axioms concern magnitudes in general.
[240] Olson (1926/27), pp. 75-76

Regarding the axioms that merely concerned geometrical magnitudes, Olson argued that they could be chosen rather arbitrarily. However, you should only use the term 'axiom' for those propositions that were not possible to derive by means of the other axioms. Still, Olson allowed informal axioms that could be proved by means of other axioms, but they appeared so self-evident to our spatial intuition that they did not need proof. The term 'axiom' should be used in connection with these self-evident propositions. In his textbook, he inserted the following proposition as a self-evident proposition that could be proved.[241]

The straight line is the shortest way (distance) between two points. [242]

3) Olson also recommended that teachers not review every proof in the textbook during the lessons. Some theorems could rather be justified by experiments or by references to symmetry. However, he did assume that all proofs were available in the textbooks. Moreover, when a teacher decided to prove a theorem during class, the proof had to be correct from a logical point of view. Olson also warned teachers that reliance on spatial intuition, when proofs were omitted, could lead to trouble. As an example he mentioned that it is very easy to base such a justification on special cases, e.g. isosceles triangles or right triangles.[243]

Just like Meyer, Olson underscored the difference between geometry at a scientific level and school geometry. An important concept in Olson's argumentation on scientific and pedagogical requirements was the concept spatial intuition. Ultimately, the quality of textbooks and teaching was in many respects dependent on whether or not they appealed to spatial intuition. Still, he did not put the requirement of spatial intuition against those of science – they did not constitute two contradictory components. He even backed up his standpoints by saying that mathematicians like Helmholtz and Peano also included the concept of motion, or movement. But, when pedagogical and scientific requirements conflicted, the former was considered more important.

The debate about textbooks and teaching methods, Part II

The second episode of the debate on elementary geometry instruction, 1938-1939, began with an article by Nyhlén where also he criticized contemporary geometry textbooks. He chose to examine the textbooks of Olson and Sjöstedt in detail. This time, too, the criticisms concerned an alleged lack of rigor; however, the faithfulness to Euclid's *Elements* was not an issue this time.

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[241] Olson (1926/27), p. 76
[242] Olson (1940), p. 4: "Den räta linjen är den kortaste vägen (avståndet) mellan två punkter."
[243] Olson (1926/27), pp. 76-78

Olson and Sjöstedt replied to Nyhlén's article, and their principal standpoint was that it would not be possible to achieve the level of rigor urged by Nyhlén in elementary instruction. Olson and Sjöstedt also criticized each other as they had different ideas on how geometry textbooks ought to be designed. Sjöstedt wanted to follow Euclid as far as possible, while Olson favoured an alternative approach. Let us first have a look at Nyhlén's criticisms.

Nyhlén's criticisms of contemporary textbooks

In a sense, Nyhlén picked up Petrini's call to teachers and textbook authors to make geometry instruction even more rigorous. However, Nyhlén did not discuss Euclid at all. The explicit purpose of Nyhlén's article was to investigate the axioms chosen by Olson and Sjöstedt. He focused on two questions.[244]

- Are the axioms independent of each other?
- Do the axioms serve as a foundation for the proofs in the so-called "congruence theory" and in the so-called "theory of parallels"?

Of these two questions, the second is the most interesting. Regarding the first question, neither Olson nor Sjöstedt disputed Nyhlén's remarks about some axioms being dependent on other axioms, i.e. some of the axioms could be proved by means of other axioms. It had not been their intention to present independent axioms, and they simply replied that the proofs proposed by Nyhlén were too complicated and that it made no sense to include them at the elementary level. The intention of the criticized axioms was to function as support for the students. The second question, on the other hand, caused a discussion on the axiomatic method and spatial intuition, which involved the whole idea of teaching axiomatic geometry.

Nyhlén's basic standpoint was the following:

> There should of course not be some state of conflict between our spatial intuitions and deductive Euclidean geometry. Our spatial intuitions should not only be employed as the axiomatic system of geometry is laid down, but should also be used in the continued construction of the system, partly as a mean to discover new theorems, partly to make the proofs more lucid. However, each new theorem must, by means of formally correct inferences, be derived from the axioms and previously proved theorems without any support of spatial intuitions.[245]

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[244] Nyhlén (1938), p.12

[245] Nyhlén (1938), p. 29: "Det får naturligtvis icke råda något slags motsatsförhållande mellan vår rumsåskådning och deduktiv euklidisk geometri. Våra rumsföreställningar skola icke endast utnyttjas vid uppställandet av axiomsystemet för geometrin utan användas vid det fortsatta uppbyggandet, dels som ett medel till upptäckande av nya satser och dels för att göra bevisen mera överskådliga. Varje ny sats måste dock medelst formellt riktiga slutledningar kunna härledas ur axiomen och förut bevisade satser utan någon som helst stöd i åskådningen."

This requirement of not relying on spatial intuitions was crucial to Nyhlén in his argumentation on rigor. He even suggested that if it was impossible to present a textbook that met this basic requirement and if proofs had to be based on spatial intuition instead, then geometry in Realskolan perhaps should be purely empirical and based on spatial intuition altogether.[246]

A basic problem with Olson's and Sjöstedt's axioms, Nyhlén argued, was that they contained concepts that needed an explicit explanation. Moreover, some important concepts were not treated in the set of axioms at all. (For Olson's and Sjöstedt's axioms, see Appendix B and C.) Thus, several proofs were useless, as they ultimately were based on spatial intuitions and not the conditions established in the axioms. However, Nyhlén did point out that the concepts of point, straight line, and plane were to be left undefined.[247] The most pressing axioms to investigate, according to Nyhlén, concerned the concept of *translation* and the so-called *geodetic property* of a straight line, i.e. the properties of the shortest line between two points, which in plane Euclidean geometry is the straight line between two points. Nyhlén's reason to consider these axioms was that they were frequently used in contemporary textbooks. If we consider textbooks of Asperén, Meyer, Olson, Sjöstedt and others, these axioms were indeed used in several proofs. Sjöstedt even claimed that the congruence theorems were a main point of the geometry courses.[248] (I exemplify how this was done in chapter N.)

Nyhlén discussed numerous examples, of which I mention only a few. We begin with the axiom on the geodetic property of a straight line. Sjöstedt did not include such an axiom, but Olson did, as did Asperén and Meyer. According to Nyhlén, Olson had established the following axiom:

> The straight line is the shortest way (distance) between two points.[249]

Actually, Olson did not include this proposition among his axioms. The reason for this was that he did not consider it a real axiom since it can be proved by means of the other properties of a straight line.[250] Nyhlén argued that a proposition like that without a proof should be considered an axiom. Since Olson had not established the meaning of *minimum* and *distance*, it was impossible to consider a proof correct when this axiom was included; such proofs were rather to be seen as empirical proofs.[251]

Nyhlén's criticisms of Olson's and Sjöstedt axioms on translations were a bit more elaborate.

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[246] Nyhlén (1938), p. 34
[247] Nyhlén (1938), pp. 22-3
[248] Sjöstedt (1936), pp. 6-7
[249] Olson (1940), p. 4: "Den räta linjen är den kortaste vägen (avståndet) mellan två punkter."
[250] Olson (1940), p. 4. Olson did not reveal how this proof should be carried out or if he referred to some theorem in Euclid's *Elements*. He might have intended theorem I.20 of the *Elements*: "In any triangle the sum of any two sides is greater than the remaining one."
[251] Nyhlén (1938), p. 13

Olson's translation axiom was formulated in the following way:

> Hereby, we assume as self-evident that lines and figures that we imagine to be changing position in space; they do not undertake any changes with respect to shape and size whatsoever.[252]

Congruency was defined in the following way by Olson:

> Two figures that in all their parts match each other and only differ with respect to position are said to be congruent.[253]

According to Nyhlén, there are two possible ways to understand the idea of a translation in Olson's axiom and both of them entailed erroneous consequences.[254]

1. We consider the translation to be a *real* translation of a material object. However, since material objects are not treated in deductive geometry, it is not allowed to make real translations. Thus, any translation needs to be *imagined*.

2. However, if the translation is *imagined*, the properties of the translation need to be established by a series of axioms. This was not done by Olson. Nor did Olson stipulate the meaning of the concepts *shape* and *size* by a definition or the other axioms. Thus, proofs based on Olson's axiom of translations are ultimately based on spatial intuition.

The critique of Sjöstedt was similar. First of all, Sjöstedt had no explicit translation axiom, which of course was a problem to Nyhlén. Another problem was Sjöstedt's axiom on congruence:

> Magnitudes that can cover each other are equal in size.[255]

Nyhlén pointed out that Sjöstedt actually used this axiom as if he meant "to cover by an appropriate *imagined* translation", see for instance the next quote below, which is taken from one of Sjöstedt's textbooks. The main problem was that the property of "to cover" was not sufficiently explained. Thus, the axiom was not possible to use in proofs since it then relied on spatial intuition.

Another problem with this axiom, Nyhlén argued, was that Sjöstedt used real translations – not imagined ones – as for instance in the proof of the first

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[252] Olson (1940), p. 15: "Härvid antaga vi såsom självklart, att linjer och figurer, som vi tänka oss ändra läge i rymden, *ej därvid undergå någon som helst förändring med avseende på form och storlek*."
[253] Olson (1940), p. 14: " Två sådana figurer som till alla delar överensstämma och endast skilja sig med avseende på läget, sägas vara kongruenta."
[254] Nyhlén (1938), pp. 9-10
[255] Sjöstedt (1936), p. 14: " Storheter, som kunna täcka varandra, äro lika stora."

congruence theorem (SAS).[256] As I see it, Nyhlén must have intended a passage like the following in Sjöstedt's textbook:

> Place ΔABC on $\Delta A'B'C'$ in such a way that the point A falls on the point A' and the side AB falls along the side $A'B'$.[257]

The wording "place … on" may be interpreted as if a real object is moved. Nyhlén also claimed that Sjöstedt must have implicitly assumed a translation axiom in connection to this proof and others.[258]

> Geometrical formations can have the same shape and size but different positions.[259]

Nyhlén had picked up this particular formulation from Sjöstedt's essay, *Geometrins axiomsystem* (The axiomatic system of geometry). In this essay, Sjöstedt did point out that "geometry requires the accuracy" of this theorem.[260]

A second problem of Sjöstedt's axioms was that it was impossible to come to a conclusion about congruence by means of the axiom on coverage and the other axioms. In the proof of the SAS-congruence theorem, it is given that two pairs of sides are equal and that the contained angles are equal, i.e. $AB=A'B'$, $AC=A'C'$ and $\angle A=\angle A'$. However, since the axioms did not establish relations between lines, planes and angles, it is not possible to draw any conclusion about a line falling along another line, Nyhlén argued.[261] It appears as if Nyhlén was referring to Hilbert's axioms of coincidence where the relations between points, straight lines and planes are established. Here, I cannot follow Nyhlén completely, but I think he means that Sjöstedt's axioms do not imply that if AB fall on $A'B'$ and $\angle A=\angle A'$, then AC will fall along $A'C'$.

In order to avoid these vague formulations about movements, placing, changing position, preservation of size and shape, etc., Nyhlén suggested a translation axiom that explicitly established the existence of congruent triangles. He exemplified this suggestion with a proof based on this axiom. Unfortunately, Nyhlén did not give a complete and definitive formulation of this axiom, but following his example quoted below and an axiom suggested by Olson it may have been formulated in the following way: *at any position*

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[256] Nyhlén (1938), p. 27
[257] Sjöstedt (1936), p. 19: "Lägg ΔABC på $\Delta A'B'C'$, så att punkten A faller på punkten A' och så att sidan AB faller utefter sidan $A'B'$."
[258] Nyhlén (1938), pp. 28-9
[259] Sjöstedt (1936/37), p. 14: "Geometriska bildningar kunna ha samma form och storlek men olika läge".
[260] Sjöstedt (1936/37), p. 14
[261] Nyhlén (1938), pp. 27-8

there exists a triangle congruent to the triangle ABC. Nyhlén proved the SAS-congruence theorem in the following way.

Given: AB=DE, AC=DF and ∧A=∧D.

Proposition: $\Delta ABC \cong \Delta DEF$

Proof: According to the translation axiom, there exist a triangle congruent to *ABC*, in which *DE* (=*AB*) is a side and where it [the triangle] is situated on the same side of *DE* as *F*. Since ∧*EDF*=∧*A*, one of the sides of the triangle falls along *DF*, and since *DF*=*AC*, *F* is the third corner of the triangle. Thus, the triangle *DEF* and the triangle congruent to *∆ABC* have all their elements equal, therefore the original triangles are congruent.[262]

The rebuttal from Olson – the educator's defense

The main argument of Olson was that Nyhlén's high demands on rigor were out of place in a discussion on textbooks intended for lower secondary level. Here, he averred that a clear separation between scientific geometry and school geometry was necessary. In the latter practice, the basic priority was that pedagogical demands should be considered more important then scientific demands for rigor.[263] Regarding the pedagogical demands, Olson referred to his previous articles in *Elementa*, published in 1926/27. I describe his standpoints in a previous chapter.

However, Olson did discuss some of the criticism leveled by Nyhlén. Regarding the choice of axioms and their interdependency, Olson did not dispute the correctness of Nyhlén's criticism from a logical point of view. But he found the criticism unjustified; an author of an elementary textbook must be able to choose some propositions as "evident starting points", without engaging in deeper thoughts on whether they are suitable as foundations for a rigorous scientific system or not.[264]

Moreover, Olson disputed Nyhlén's accusation of his proof of the SAS congruence theorem being less rigorous because of an inadequate axiom of translation. Olson did not consider Nyhlén's alternative proof above to be more rigorous; it was just more abstract and less tangible to the students. The axiom used by Olson in his textbook was the following:

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[262] Nyhlén (1938), p. 18: "F: *AB=DE*, *AC=DF* and ∧*A*=∧*D*.   P: *∆ABC ≅ ∆D*
B: Enligt flyttningsaxiomet existerar en triangel kongruent med *ABC*, i vilken *DE* (=*AB*) är en sida och som är belägen på samma sida om *DE* som *F* [sic!]. Emedan ∧*EDF*=∧*A*, faller då en sida av triangeln utefter *DF*, och då *DF*=*AC*, så är *F* tredje hörnet i triangeln. Triangeln *DEF* och den med *∆ABC* kongruenta triangeln ha alltså alla element lika, varav följer, att de ursprungliga trianglarna äro kongruenta."
[263] Olson (1938), p. 94
[264] Olson (1938), p. 94

Hereby, we assume as self-evident that lines and figures that we imagine to be changing position in space do not undertake any changes with respect to shape and size whatsoever.[265]

Olson pointed out that this was a more concrete version of the axiom.

To a given diagram there is a corresponding diagram at an arbitrary position in space that accords with the latter with respect to its different elements.[266]

Olson's reason for choosing the former axiom was that it provides a more tangible explanation about the existence of a second congruent triangle.[267] The example Olson considered is the proof of the first congruence theorem. Here one has to prove that two triangles *ABC* and *DEF* are congruent if *AB=DE* and *BC=EF* and $\angle B = \angle E$. Olson points out that if we use the latter axiom or the axiom suggested by Nyhlén in the proof, we then conclude that there exists a third triangle congruent to the triangle *ABC* lying on the triangle *DEF*. According to Olson, it is more convenient to say that we move the triangle *ABC*, even though it is not a real translation.[268]

Olson did not bother to comment on Nyhlén's criticisms regarding the undefined concepts shape and size.

An important aspect of Olson's reply is that he carefully underscored the difference between school geometry and geometry at a scientific level; the textbook author and the mathematician are doing different things that cannot be measured by the same standards. On the basis of this distinction, he avoids much of the criticism by saying that Nyhlén had not acknowledged that circumstance. We can compare this with Sjöstedt's rebuttal, which is quite different in this respect.

The rebuttal from Sjöstedt[269] – the philosopher's defense

Sjöstedt also accentuated the difference between school geometry and scientific geometry.

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[265] Olson (1940), p. 15: "Härvid antaga vi såsom självklart, att linjer och figurer, som vi tänka oss ändra läge i rymden, *ej därvid undergå någon som helst förändring med avseende på form och storlek*."

[266] Olson (1938), p. 85: "Mot en given figur svarar på en godtycklig plats i rummet en annan figur, som fullt överensstämmer med den förra med avseende på sina olika element."

[267] Olson (1938), p. 94

[268] Olson (1938), pp. 85, 94

[269] Sjöstedt reply was published in the journal *Elementa* in 1938. However, the same year he also published three consecutive articles in the journal *Tidning för Sveriges läroverk*. These articles were titled "Realskolans geometriundervisning" (~ Geometry instruction in Realskolan). In principle, he conveyed the same standpoints in these articles as in his reply to Nyhlén in *Elementa*.[Sjöstedt (1938b)]

When you discuss the basic concepts and the axioms of geometry, as I see it, you then have to make a sharp distinction between that which belongs to scientific geometry and that which can fit school geometry.[270]

In contrast to Olson's reply, though, Sjöstedt did not dismiss Nyhlén's criticism as being too scientific or misplaced in a pedagogical context. Instead, he dealt with Nyhlén's idea of geometry and axiomatic systems.

According to Sjöstedt, the function of axiomatics could be described in the following way:

To erect a system of propositions that are necessary and sufficient to make the system valid for just a certain group (or certain groups) of concepts.[271]

Sjöstedt's standpoint was that the interpretation of this function was often faulty. The error was that it was believed that the axioms determined the concepts. This belief was based on the conception that an increasing number of axioms ruled out any other concepts. Sjöstedt found this conception unreasonable. If you wanted to formulate axioms that are independent, but at the same time necessary and sufficient, it was not possible to completely separate a certain set of concepts, e.g. point, straight line, and plane, from other concepts. In turn, he argued, that particular circumstance often led to the misconception that the concepts are undetermined.[272]

Sjöstedt argued that the geometrical concepts, in Euclidean geometry, that is, are completely determined, but not via axioms, nor via experience. The crucial source in this matter was our spatial intuition or our conception of space.

I have now tried to show ... that the geometrical concepts are fully determined, which they have to be, even though it is not possible to define the basic concepts of point, straight line, and plane in a proper manner. However, you know what is designated when these names are used. Therefore, there can be no further amendments to the determination of the concepts in question, neither through experience nor through an axiomatic system. In general, whether the concept in undefined or not, we cannot say that a group of axioms defines a concept. If you do not know what congruence means, for instance, then Hilbert's axiom of congruence loses all meaning.[273]

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[270] Sjöstedt (1938a), p. 165: "Då man diskuterar geometrins grundbegrepp och axiom, måste man enligt min mening hålla skarpt isär, vad som hör till den vetenskapliga geometrien och vad som kan passa i skolgeometrien."

[271] Sjöstedt (1938a), p. 167: " Att uppställa ett system av satser, som äro nödvändiga och tillräckliga för att systemet skall gälla endast för en viss grupp (resp. vissa grupper) av begrepp."

[272] Sjöstedt (1938a), pp. 167-168

[273] Sjöstedt (1938a), p. 166: "Jag har nu sökt visa ... att de geometriska begreppen äro fullt bestämda och måste vara det, även om de grundläggande begreppen punkt, rät linje och plan icke kunna i egentlig mening definieras. Man vet dock, vad som avses, då dessa termer används. Ifrågavarande begrepp kunna därför icke på något sätt erhålla något tillägg till sin

Even though he did not mention "conception of space" in this passage, he did make this connection in a later section.

> It is true that the inferences in a geometrical proof must be formally correct, i.e. each proposition which is not an axiom shall be obtained as a conclusion with other propositions as premises. Thus, the propositions can ultimately be traced back to the axioms, which, according to my opinion, are characterized by being the ultimate conditions of geometry with respect to propositions. But, a geometrical proof contains something beyond these propositions, namely our conceptions of space. These are not propositions and can therefore not be traced back to the axioms. Likewise, they are necessary conditions of a proof. Without the realization of our conceptions of space, you cannot accomplish a geometrical proof.[274]

In accordance to this view, all the concepts in Euclidean geometry were determined via our spatial intuition. Consequently, Sjöstedt found Nyhlén criticisms about giving insufficient definitions or axioms about congruence unjustified.

> Lecturer N. thinks that the regular definition of congruent figures (i.e. figures with the same shape and size but with different positions) is unsatisfactory, because what is meant by shape and size? A definition of these general concepts ought to be impossible. But, does this entail that these concepts are undetermined? That would be the same type of confusion as with the definitions and determination of the basic geometrical concepts. The point, the straight line, and the plane cannot be defined in any rigorous manner, but still they are fully determined. The same applies for the concepts shape and size. If one says that two figures have the same shape and size or that they have the same shape but different size, then Do we not know what is designated? Therefore, I cannot consider the definition just mentioned in any way unsatisfactory, especially not in a textbook at Realskolan. How then should the concept be defined?[275]

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bestämdhet, vare sig genom erfarenhet eller genom axiomsystem. Överhuvud kan en grupp axiom icke sägas definiera ett begrepp, vare sig begreppet är odefinierat eller ej. ... Vet man t. ex. icke, vad kongruens innebär, så förlora Hilberts kongruensaxiom varje mening."

[274] Sjöstedt (1938a), pp. 168-169: "Det är visserligen sant, att de slutledningar ett geometriskt bevis innehåller, måste vara formellt riktiga, d.v.s. varje ingående omdöme, som icke är ett axiom, skall ha erhållits som slutsats med andra omdömen såsom premisser. I sista hand skola alltså omdömena återföras på axiomen, som enligt min mening just karakteriseras av att vara geometrins yttersta förutsättningar av omdömeskaraktär. Men ett geometriskt bevis innehåller något utöver dessa omdömen, nämligen vissa rumsföreställningar. Dessa äro icke omdömen, kunna därför icke återföras på axiomen. Men de äro likväl nödvändiga förutsättningar för bevisföringen. Utan att man realiserar rumsföreställningar kan man överhuvud icke föra ett geometriskt bevis."

[275] Sjöstedt (1938a), p. 174: " Lektor N. anser den vanliga definitionen av kongruenta figurer (såsom figurer med samma form och storlek men olika läge) vara otillfredsställande, ty vad menas med form och storlek? Att definiera dessa allmänna begrepp torde vara omöjligt. Men äro därmed dessa begrepp obestämda? Det vore samma förväxling av odefinierbarhet och obestämdhet som vid de geometriska grundbegreppen. Punkt, rät linje och plan kunna ej i någon sträng mening definieras men äro dock fullt bestämda. Detsamma gäller om begreppen

Even though Sjöstedt did not state it explicitly, we understand that he authored his textbook in accordance with this view.

Another aspect of Sjösted's reply to Nyhlén's article is that it indirectly allowed a more lucid wording.

> Regarding the textbooks at Realskolan, the logical point of view may never, according to my opinion, be satisfied at the expense of the pedagogical one. In this context, I just want to mention one example. An expression such as "how far they may be extended" in the ordinary definition of parallel lines is of course unnecessary from a logical point of view. But is it then unnecessary or inappropriate in a school book? I think so not.[276]

See also the quote below where Sjöstedt argues that it is all right to talk about triangles being moved.

Of course, Sjöstedt recognized that Nyhlén did not share his view;[277] to Nyhlén, any kind of axiomatic geometry should be completely freed from spatial intuition in all instances. Sjöstedt's argument as to why his view and his textbooks are more appropriate was the following. To a mathematician a more formalistic system is interesting since it applies to other sets of concepts than point, straight line, and plane. But in schools, he considered Hilbert's axiomatic system inappropriate. He did not say why, but he probably found it too abstract.

Regarding Nyhlén's criticism of the axiom on movements, Sjöstedt admitted that he indeed had claimed that geometry requires the correctness of the proposition …

> Geometrical formations can have the same shape and size but different positions.[278]

In this respect, Nyhlén was perfectly right. However, he had failed to notice or he had ignored Sjöstedt's attempt to show that this proposition is not an axiom since it is not used a premise in the proofs. Thus, there is no need to use the proposition on translation as an axiom, since it is not an axiom, Sjöstedt argued.

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form och storlek. Om man säger, att två figurer ha samma form och storlek eller att de ha samma form men olika storlek, nog vet man, vad som avses? Jag kan därför icke anse den nämnda på något sätt otillfredsställande, allra minst i en lärobok för realskolan. Hur skulle man här eljest definiera begreppet?"

[276] Sjöstedt (1938a), p. 165: " Då det gäller en lärobok för realskolan, får enligt min mening den logiska synpunkten aldrig tillgodoses på bekostnad av den pedagogiska. I detta sammanhang vill jag nämna endast ett exempel. Ett uttryck sådant som >>hur långt de än utdragas>> i den vanliga definitionen av begreppet parallella linjer är givetvis från logisk synpunkt onödig. Men är därför onödigt eller olämpligt i en skolbok? Jag tror det icke."

[277] Sjöstedt (1938a), p. 168

[278] Sjöstedt (1936/37), p. 14: "Geometriska bildningar kunna ha samma form och storlek men olika läge".

That the size and shape of a diagram are not altered by a change in position is a precondition of the same nature as the precondition that a diagram is not altered, but maintain its properties. To my knowledge, nobody has proposed the idea to submit the latter proposition as an axiom. But, it is as necessary as the "translation axiom." I cannot accept the objection of lecturer N. that I should have forgotten that only thought translations are allowed. It is obvious that real translations only occur in connection to physical solids. But, is it each time really necessary to emphasize that the translation of the diagrams are *imagined*? Can any nuisance occur if one ignores this? In principle, my proof of the first congruence theorem is identical with the proof submitted by lecturer N. on page 18. The only difference is that I do not quote the "translation theorem" as a premise since it is not used as such. One thinks of triangle congruent with ΔABC, but in another position. That is necessary. But, it is not necessary to pass the judgment about the possibility of doing it.[279]

Sjöstedt's reply to the criticisms on the dependency of the axioms was that Nyhlén had missed the point of these axioms. Sjöstedt used two types of axioms in his textbooks: geometrical axioms and general axioms on magnitudes. It was the latter Nyhlén criticized for being dependent. These should, however, just be seen as pedagogical support for the students, Sjöstedt argued; they did not function as real axioms in an axiomatic system. Moreover, from the very beginning, Sjöstedt had considered these axioms to be tautological. Thus, there was no point in giving proofs for some of these axioms.[280]

In many ways, the Sjöstedt's argumentation appears to be a lengthier and more intricate version of Olson's arguments, but in principle saying the same thing: school geometry and scientific geometry are different and cannot be measured by the same standards. However, this is not entirely correct; a crucial difference is that Sjöstedt provided an explanation for why school geometry can be given the form he had given it and at the same time not be considered unscientific.

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[279] Sjöstedt (1938/39), pp. 174-175: "Att en figurs storlek och form icke ändras vid en läge-ändring är en förutsättning av samma natur som förutsättningen, att en figur överhuvud icke ändras utan bibehåller sina egenskaper. Ingen har mig veterligt kommit på idén att anföra den senare satsen som axiom. Men den är ju lika nödvändig som >>flyttningsaxiomet>>. Lektor N:s invändning, att jag skulle ha glömt bort, att man endast får tänka sig flyttningen, kan jag inte godtaga. Det är ju klart, att verklig förflyttning endast förekommer vid fysiska kroppar. Men behöver man verkligen ideligen betona, att man *tänker* figurerna flyttade? Kan någon olägenhet uppstå, om man underlåter det? I princip är mitt bevis för första kongruensfallet identiskt med det bevis lektor N. anför s. 18. Den enda skillnaden i sak är, att jag icke citerar >>flyttningsaxiomet>> som premiss emedan det faktiskt icke användes som sådan. Man tänker en med ΔABC kongruent triangel i ett visst annat läge. Det är nödvändigt. Men det är icke nödvändigt att fälla omdömet, att man kan göra det."
[280] Sjöstedt (1938a), pp. 171-173

Differences between Olson and Sjöstedt – To follow Euclid or not to follow Euclid

Olson and Sjöstedt had quite different views on how a textbook ought to be designed. Olson did not hesitate to deviate from the classical Euclidean outline, and he inserted theorems about symmetry, theorems that on some occasions were used instead of the Euclidean congruence theorems. Sjöstedt, on the other hand, wanted to keep the Euclidean structure as far as possible in his textbooks, even though he did make alterations. A statement in the foreword of Sjöstedt's textbook sums up his position on Euclid's *Elements* and textbooks.

> The question regarding whether Euclid's or some other course should be used in connection with basic geometry instruction is an old matter of dispute. The undersigned has gradually gained an increasingly confirmed conviction that Euclid's course is superior in its entirety. None of the attempts to build geometry differently, which I have been informed about, do I consider even equal to the old man's. These attempts are also, as I see it, more successful the more they agree with Euclid's course. This does not, however, prevent me from considering certain modifications of Euclid's system to be necessary …[281]

Sjöstedt identified three types of modifications. 1) One should keep the essential parts of Euclid's course, but the justified pedagogical demands were not be set a side. 2) Due to the scientific investigations on the foundations of geometry from the preceding 100 years, Euclid's axiomatic system should be modified. 3) The courses in geometry should not be too extensive, but more focused on the essential parts.[282]

According to the foreword of Sjösted's textbook, the main modifications were the following. Euclid's congruence theorems were an essential part of the course. They were also pedagogically sound since they offered clarity and order, and they were possible to apply from the very beginning of the courses an onwards. Consequently, they were kept. Theorem I.16 in Euclid's *Elements*, on the other hand, was cancelled for pedagogical reasons, even though it provided scientifically interesting proofs that were not based on the postulate on parallel lines.[283] Sjöstedt did not specify what these pedagogical reasons were, but the geometry exercises in the final exams offer a clue. In order to solve these exercises, you do not have to apply the Theorems I.16 to

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[281] Sjöstedt (1936), p. 5: "Frågan, huruvida man skall använda Euklides' eller någon annan lärogång vid den grundläggande geometriundervisningen, är en gammal stridsfråga. Undertecknad har så småningom vunnit en alltmer befästad övertygelse, att Euklides' lärogång i det hela är överlägsen. Intet av de försök jag tagit del av, att uppbygga geometrien på något annat sätt kan jag anse ens jämbördigt med den gamles. Dessa försök äro också enligt min mening desto mer lyckade, ju mer de överenstämma med Euklides' lärogång. Detta hindrar emellertid icke, att jag anser vissa modifikationer i Euklides' system vara nödvändiga, ..."
[282] Sjöstedt (1936), pp. 5-6
[283] Sjöstedt (1936), p. 7

I.25 in Euclid's *Elements*. Indeed he cancelled all these theorems except for I.22 and I.23.[284]

Due to these cancellations he had to provide a new proof of the ASA-congruence theorem and the theorems on parallel lines. I return to these proofs in the next part of this dissertation. One of the modifications that followed upon scientific considerations was the insertion of an axiom on the continuity of lines.[285]

In many ways, Sjöstedt followed Petrini's appeal to develop Euclid's *Elements*. He retained the Euclidean congruence theorem and he added axioms that were missing in the *Elements*. However, he was much more willing to modify the order of the theorems, and he also added new theorems. I return to this issue in the next part of the dissertation.

Concluding remarks – Professional debates about elementary geometry instruction

Geometry instruction in Realskolan

Arguments about contents

The curricula do not reveal much about the contents of the geometry courses, but if we consider the textbooks, they all contain a core of theorems from book I and III of Euclid's *Elements*.[286] In the professional debate, the content of the courses was not really called into question. Even though Euclid's *Elements* and textbooks that followed Euclid very closely were criticized for being unsuitable as textbooks, there were no discussions about doing a complete make-over of the geometry courses. There were no arguments about replacing the existing content by elements from analytical geometry or vector geometry. This implies that there was some sort of consensus regarding the content of the courses.

Not until 1938 was there a suggestion to give up the axiomatic method. One of the debaters, Nyhlén, argued that if the proofs did not meet the scientific standards of a proof, then one should found the elementary geometry on experiments and inductive reasoning.

The main disagreements concerned methodological issues rather, i.e. the order in which the theorems should be presented, how the proofs should be

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[284] I.22: Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any matter should be greater than the remaining one. [Heath (1956), p. 292]
I.23: To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it. [Heath (1956), p. 294]
[285] Sjöstedt (1936), pp. 7-8
[286] See Part D of the thesis for further details.

designed and the level of rigor in the proofs. I will return to these issues in a moment.

Arguments about goals

In the debate, the main goal of geometry instruction was to provide training in reasoning. Especially, the courses in axiomatic geometry were linked to this goal. The goal was first and foremost formulated in textbooks and articles about geometry instruction. Not until 1955 was a formulation about this goal inserted in the curriculum. In the previous curricula for mathematics it was established that students were supposed to attain skills and knowledge that were useful in every day life and working life. There were also formulations about providing a general civic education. Hence, here we can observe a discrepancy between the curricula and the arguments of the elite.

The goal about training in reasoning did of course encompass abilities to master logic and the axiomatic method, but there were other components as well. Let us first consider the epistemological aspects. In the debate, the debaters, especially Meyer, Hedström, Olson, and Sjöstedt, underscored that definitions, axioms, theorems, and proofs had to appeal to the students' spatial intuitions. Thus, the goal of geometry instruction was not simply a matter of attaining skills in logic. However, the arguments about spatial intuition mainly concerned methodological issues.

One debater, Olson, also underscored the value of getting the students to understand why the axiomatic method was necessary and a more effective method than inductive or experimental methods; if the students did not understand why the proofs were necessary, they would not be motivated to learn. Hence, he was pointing to another type of knowledge: knowledge about logic and the axiomatic method was something different than knowing how to carry out a proof.

Apart from the epistemological goals, several debaters took aim at the moral aspects of geometry instruction and training in reasoning. In this context, critical thinking was the unifying idea. One argument was that geometry instruction fostered independent reasoning. A second argument was that geometry instruction conveyed a general ability to handle language. A third argument was that geometry instruction and the axiomatic method made the students aware of their spatial intuitions; geometry instruction sharpened the students' conception of space. It is worth noting that there were no arguments about the preservation of some historical or cultural heritage. The sole aim was the individual's ability to reason.

As I have mentioned, the curriculum did not contain an explicit formulation about training in reasoning until 1955. But that does not entail that the debaters contradicted the curriculum. As I see it, they gave a more precise meaning to the goal about conveying a general civic education in connection to mathematics instruction, a goal that was explicitly stated in the curricula.

Taken together, the arguments described above give us a picture of what the expression 'training in reasoning,' or simply 'reasoning,' could mean in connection with geometry instruction. The expression 'training in reasoning' had several connotations, and it was not just a question of learning logic and proofs.

Thus far, I have described the goals and the aspects of training in reasoning the debaters wrote about. However, there are aspects of geometry instruction that the debaters did not discuss in terms of aims or training in reasoning. One essential aspect that the debaters did not discuss is the heuristics of geometry or questions regarding discovery.[287] A concrete goal in connection to elementary geometry instruction could have been to become a skilled problem solver. The arguments about spatial intuition may be seen as a heuristic element, but spatial intuition was primarily mentioned in connection with the understanding of concepts, theorems, and proofs. Spatial intuition should function as a support for the proofs.

As a matter of fact, such a discussion on heuristics would not have been inappropriate. In part E of this dissertation, I show that the geometry exercises in the final exams required not only an ability to solve rather complicated problems, but also an ability to compose short proofs on your own. The geometry exercises of that type were included in each final exam throughout the period 1905-1962. In this part E, I also argue that it was more important for the students to solve problems than to master techniques of proof in order to pass the geometry exercises at the exam test.

Moreover, at the end of the period, contemporary literature on the subject existed. Pólya's first work on problem solving, *How to solve it*, was published in 1945. Here, problem solving was given a thorough treatment and Polya offered strategies for problem solving.[288] Actually, Sjöstedt published a book where he discussed methods to solve geometrical problems.[289] However, this textbook was primarily intended for upper secondary schools and he did not bring up problem solving in the debate on geometry instruction.

My point here is that the relevance of the arguments in the professional debate was limited to issues related to the axiomatic method, proofs and textbook.

However, these arguments were by no means irrelevant since the textbooks were designed in accordance with the axiomatic method. Moreover, there were indeed alternatives to choose from, which I show in Part D of the thesis.

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[287] One exeption in this respect is Olson, who mentioned problem solving in one of his articles. However, he did just mention it briefly. [Olson (1926/27), p. 82]

[288] Polya (1970). Not until in 1970, Polya's book on problem solving was published in Swedish.

[289] *Geometriska övningsuppgifter och lösningsmetoder*, published in 1932, 1936, 1948 and 1960.

Arguments about methodologies

In the debate on the courses in axiomatic geometry in Realskolan, the argumentation about how proofs and theorems should be designed and introduced followed three lines. Moreover, the argumentation very much concerned the concepts of rigor and spatial intuition.

One standpoint was that reasoning based on spatial intuition would muddle the proofs. According to the proponents of this argument, the textbook authors should aim at proofs that did not rely on spatial intuition; each inference in a proof had to be based on previous theorems alone. On this point, they were very critical about the common use of translations or movements of geometrical objects. These standpoints were primarily promulgated by Petrini some years before 1920 and in the 1920's and by Nyhlén in the late 1930's. Petrini, also argued that Euclid's *Elements* was the role model in this respect. However, the *Elements* could be improved, he argued. Nyhlén, on the other hand, did not mention the benefits of Euclid's *Elements*.

A second standpoint was that the understanding of geometry required more than logic; the proponents of this argument argued that concepts, theorems, and proofs should be introduced in way that appealed to students' spatial intuition. They also underscored that it was not possible to apply the same level of rigor in school geometry as in scientific contexts. This standpoint was primarily promulgated by Meyer, Hedström and Olson mainly in the 1920's and by Olson in the late 1930's. One of the debaters, Meyer, specified the arguments about spatial intuition. Proof should be based on concepts of foldings and symmetry. As I show in the next part of this dissertation, textbook authors like Asperén and Olson used such proofs. On the basis of these arguments, Meyer found Euclid's *Elements* inappropriate as a textbook at the elementary level.

A third standpoint, promulgated by Sjöstedt in the late 1930's, was that one should follow Euclid's course as far as possible. But he did allow a wording that is closer to everyday language. For instance, in his textbooks he used formulations where triangles were moved as if they were real objects. He thought this was more functional in a school book than referring to some abstract translation theorem about the existence of congruent triangles. Moreover, he insisted that concepts and theorems had to appeal to our spatial intuitions.

Apart from these standpoints regarding the design of textbooks, one methodological issue that was put forward in the discussions concerned the justification for using the axiomatic method. Some debaters, Meyer and Olson, underscored the value of exercises or discussions where the students could discover the advantage of applying the exact axiomatic method instead of less exact experimental and inductive methods.

An important aspect of the methodological arguments is that, just like the arguments about the goals, they mainly concerned the axiomatic method, i.e.

the choice of axioms, the design of proofs, the disposition of theorems, the level of rigor, and the proofs' connection to spatial intuition. However, the students were not overlooked in these discussions. The argumentation also included the students' understanding and appreciation of the axiomatic method and the proofs; some argued that the proof should appeal to the students' spatial intuition; others argued that the teachers should motivate the students by emphasizing the value of the axiomatic method. Also Petrini included discussed the needs of the students; he claimed that a high level of rigor helped the students to understand better. Consequently, we cannot say that the discussions on the axiomatic method were purely scientific and lacked relevance from a pedagogical point of view.

Regarding the relevance of the methodological arguments about the axiomatic method and textbooks design, I would say that they were relevant also to the common teachers since there were different types of textbooks to choose from during the period 1905-1962. Nonetheless, just as the heuristic aspects were not a part of the discussions on the goals of geometry instruction, they were not either part of the discussions on methodologies. Spatial intuition was a central concept in the methodological discussions, but it was not part of a discussion on how the students should learn how to solve problems. Thus, the relevance of the methodological arguments was also restricted to the axiomatic method.

Geometry instruction in Folkskolan

Arguments about contents

As I have pointed out, my sources for the chapters on the debate on geometry instruction in Folkskolan has been literature used at teacher training institutes. This might be a reason why the contents of the courses were not called into question and discussed. Such questions were not for the lecturers and teacher trainees to discuss or decide upon. Moreover, the curriculum of 1919 was very clear that Folkskolan was to provide skills and knowledge in mathematics useful in every day life and working life. What is more, all the investigated textbooks contained a lot of exercises were the students were supposed to calculate lengths, areas, and volumes. My point is that there were no major ambiguities about what should be the core of the geometry courses.

Arguments about goals

If we consider the curriculum for Folkskolan of 1919, the goals of mathematics instruction, geometry included, were to provide skills and knowledge that were useful in every day life and working life. There were no formulations about some general civic education; such formulations appeared only

in the curriculum for Realskolan. Nor were there any formulations about training in reasoning.

The goal regarding training in reasoning in connection with mathematics instruction was not an exclusive concern for Realskolan, however. It occurred in the literature used in teacher training as well. As early as the 1925, Wigforss stated that mathematics instruction in Folkskolan should provide training in reasoning. Setterberg had explicated the same goal some years earlier. However, they made no special references to geometry instruction.

Actually, Wigforss did mention the heuristic aspect of studies in mathematics, but just briefly.

The impact of the argument about training in reasoning may not have been that great when Setterberg's and Wigforss' books on mathematics instruction were published about 1920. The books were only used at two institutes for teacher training during the 1920's and 1930's – the two institutes where Setterberg and Wigforss worked as lecturers. However, during the 1940's, Wigforss' book was used at an increasing number of institutes. In 1955, the goal about training in reasoning in connection to mathematics instruction was inserted in the curriculum of Folkskolan. The formulations about skills and knowledge useful in everyday life and working life even took on a less prominent position. Also in the curriculum there were no special references to geometry instruction in connection with the formulations about training in reasoning.

However, if we relate the two goals to the textbooks and the chapters on geometry, it is the goal of providing skills and knowledge useful in everyday life and working life that appears to have been the most relevant to the textbook authors. Throughout the period 1905-1962, the textbooks investigated were dominated by exercises where the students were supposed to compute lengths, areas, or volumes of geometrical objects. Oftentimes, the exercises had some practical connotations as well.

Of course, such a procedure requires some reasoning and I do not deny that these types of exercises can be difficult. In the next part of the thesis, I will describe the textbooks and the exercises, but, for now, suffice it to say that a solution to these exercises in applied geometry followed a standard procedure: first you choose a formula, then you plug in the numbers in the right positions, and finally you make a calculation. In contrast, some textbooks did contain other types of exercises where the solution was a bit more complex and not a matter of applying such a standard procedure. Still, these exercises were very rare. Considering the complexity of the exercises, the goal about training in reasoning appears to have been quite irrelevant as the textbooks authors designed the chapters on geometry.

If we then compare the texts intended for the teachers in Realskolan, on one hand, and the texts intended for the teachers in Folkskolan, on the other, we can observe that the expression 'training in reasoning' had different meanings. In the former texts, training in reasoning was closely connected

with the courses in axiomatic geometry; in the latter texts, training in reasoning had a wider meaning and the expression was not restricted to geometry.

Obviously, one has to remember that we are comparing two different school types, where Folkskolan did not have courses in axiomatic geometry and Realskolan had older students, at least for most of the period 1905-1962. However, in the 1950's, Folkskolan could comprise up to 9 classes in some cities and municipalities and the curriculum in mathematics was extended, with more advanced mathematics, though not axiomatic geometry. Moreover, the 1950's was a period when the curriculum of the coming Grundskolan was being prepared and the curricula of Folkskolan and Realskolan were about to merge into one. In that context, I think it is interesting that training in reasoning could mean different things.

Arguments about methodologies

In the curricula of 1919 and 1955 it was established that visualizability, i.e. åskådlighet, should be the leading principle in connection to mathematics instruction, geometry included. In the teaching literature used in teacher training, these directives were explained in more detail, especially by Wigforss. The basic argument was that a teaching method based on visualizability should counter mindless learning by rote. A central component of this method was that students should be active in making observations or manipulations of illustrations or other objects; actions that were supposed to draw attention to the essential features of a concept or a proposition.

An important detail in Wigforss' argumentation is that he linked training in reasoning to visualizability and this type of activity.

According to the teaching literature, this teaching method also included a routine for how concepts, formulas, and other propositions should to be introduced and explained. Experimental exercises were supposed to stimulate the students to make observations and manipulations; at the end of a sequence of such exercises, a definition, a formula, or some other proposition was established. If we consider the textbooks of the period 1905-1962 and the chapters on geometry, this routine was applied by all authors of the textbooks investigated. (The textbooks are described in the next part of the thesis.) Of course, there were differences, and the textbooks did change over time, but the experimental routine remained in some form.

Thus, it appears as if the arguments about training in reasoning were relevant to the textbook authors, but only when they designed introductions and explanations of concepts, formulas and other propositions.

An important aspect of the arguments about the experimental routine is that it appeared in geometry textbooks well before Setterbergs and Wigforss works on mathematics instruction. Thus, their arguments about teaching methods based on spatial intuition were relevant, but hardly innovative. In this particular issue, Wigforss rather reinforced existing ideas on teaching methods and textbook designs. In comparison to Setterberg's and others

treatises, Wigforss also added some more elaborate formulations about outer and inner visualizability. However, he did not give any further explanations about what outer and inner visualizability meant.

To end with, I would like to emphasize that the argumentation about teaching based on visualizability contained much more than showing illustrations; it had much wider connotations than that.

The arguments and the status of the debaters

If we consider the sources that I have used for the investigation of the debate on geometry instruction in Folkskolan, the relation between the arguments and the authors' professional status is quite unambiguous. The texts were used in teacher training and standpoints were justified but not criticized. Hence, what we see here is a direct link between the professional elite and the teacher trainees that was not interfered with by critics.

In the investigated debate on geometry instruction in Realskolan, the situation was quite different. Here, arguments were countered and the debaters presented different alternatives regarding geometry instruction. However, this debate was not exactly balanced if we consider the status of the debaters. It is possible to distinguish a center and a periphery if we by center mean positions with the central school authorities, editors of journals, involvement in teacher training, and authors of the most popular textbooks.

In the center then, we find Meyer, Hedström, Olson, and Sjöstedt, on the periphery, Petrini and Nyhlén. Not only that, the arguments of the persons in each group also shared some basic features, even thought there were clear differences as well. The common feature of the former groups is that they were careful about making a clear difference between pedagogical and scientific demands. Moreover, when these demands conflicted, the former should be given precedence. These pedagogical requirements were often linked to spatial intuition. Not that proofs and the axiomatic method should be replaced, but concepts, theorems and proofs should appeal to spatial intuition in some way. Petrini and Nyhlén, on the other hand, did not accentuate a difference between pedagogical and scientific demands. Moreover, their criticisms about the prevalent circumstances were primarily delivered from a scientific point of view.

The existence of a center and a periphery adds an extra dimension to the professional debate, especially if we consider the debate as an incentive for actions and a source of arguments. According to my investigations, it appears as if the actors involved in mathematics instruction had to acknowledge a clear difference between pedagogical and scientific demands in order to make a career alongside of teaching. Moreover, you had to recognize and appreciate the value of spatial intuition as well.

Part D – Geometry textbooks in Folkskolan

The supply of textbooks

Before the new curriculum for Folkskolan in 1919, geometry and arithmetic constituted two subjects. This might be one reason to why there were separate textbooks for each subject. As the two subject merged into one, only textbooks containing both arithmetic and geometry were written. However, the publishers kept on printing separate textbooks for geometry as late as the early 1950's, but some combined textbooks were also written before 1919. I denote these two types of textbooks *separated textbooks* and *combined textbooks*. One difference between these two kinds of textbooks is that the combined ones were divided into series of books for each grade, while the separated ones encompassed all geometry courses in Folkskolan. Another difference is the number of pages devoted to geometry. The combined books comprise in total about 45 pages and the separated about 65.[290]

I have investigated the following textbooks:

Table 5.

author	separate					combined, series								
	Lundborg	Danielson	Ohlander	Knutsson	Dalin	Hellsten et al	Lövgren et al	Roman et al	Rydén et al	Nord	Asperén et al	Kärrlander	Sandström et al	Lindström et al
first printing	1896	1910	1914	1914	1922	~1921	1926	~1927	~1928	1929	1930	1930	~1934	~1944
last printing	1952	1942	1936	1943	1950	~1943	~1950	~1950	~1958	-	1953	-	~1956	~1952
# editions	20	25	9	10	2	2 or 3	~10	5 or 6	8	1	8	1	~12	5

Some years are imprecise (indicated by ~) since the whole series were not reprinted in the same year. The imprecise number of editions is because sometimes only parts of the series were being reprinted.

According to a survey done in the early 1930's, there were 99 different textbooks in arithmetic and geometry for Folkskolan and so-called higher Folk-

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[290] In general, the textbook authors did comply with the directive of the curricula; yet, the separated textbooks had a more extensive course in year 4 in comparison with the directives of the 1919 curriculum. The separated textbooks investigated in this study were first published before 1922, and the combined textbooks were published after 1921.

skolor. Some editions could comprise about one million books, while some were used only by their respective authors.[291] By the time of the first official list of approved textbooks issued by the State textbook agency, i.e. Statens läroboksnämnd, in 1940, the number of textbooks for Folkskolan and higher Folkskolor was reduced to 63. On that list, all of the textbooks in the table above were included, except for Lindström et al, which had not been published yet.[292] On the list of 1955, the number of approved textbooks was reduced to 34. Here, all textbooks in the table above but those by Roman & Wigforss, Rydén, Frank & Norgren and Lindström, Jonzon & Jansson had been cancelled. Sandström & Jonsson, who were cancelled from the list, published a new version of their textbook that was accepted. Moreover, Lindström, Roman, and Wigforss were involved in the authoring of some other textbooks during the 1950's. Apart from the textbooks written by these authors, there were only two other textbooks, after 1955, that covered the fourth year and onwards in Folkskolan.[293]

Different approaches to visualizability and self-activity

A common feature of all the textbooks investigated is that the authors applied some kind of approach based on *visualizability* and *self-activity*. In several of the forewords of textbooks, the concepts *visualizability* or *self-activity* or both together were mentioned; according to the authors, the books were written in such a way that they promoted *visualizability* or *self-activity*. Moreover, on various occasions, the students were supposed to observe or measure an illustration or some other real object, e.g. a pencil box, a book, or the desktop.[294] Another common feature of the textbooks investigated is that introductions and explanations of concepts, formulas, and other propositions followed a certain routine; the students were supposed to read or work through experimental exercises before definitions and propositions were explicitly stated. Hence, concepts were not introduced by a list of definitions; the introductions of formulas and other propositions were not simply a matter of stating them.

However, these experimental introductions and explanations were designed in different ways. In the separate textbooks and some of the combined textbooks, i.e. the older books, the authors had submitted written lines of

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[291] Article in *Folkskollärarnas tidning* (1934), nr. 37, p. 37
[292] Statens läroboksnämnd (1940), pp. 90-97
[293] Statens läroboksnämnd (1955), pp. 45-52
[294] Asperén et al (1931), p. 3; Hellsten et al (1938), first page; Nord (1929), p. 2; Hoffstedt (1940), p. 3; Knutsson (1922), p. 2; Kärrlander (1930a), p. 3 and Lindström (1946), on the inside of the cover. Even though Dalin (1923) and Danielson (1925) do not make any comments on *åskådlighet* and self-activity, but every section in their books starts with a description of materials needed to facilitate *intuition*.

thinking where the reader is guided through the experimental exercises.[295] In the newer combined textbooks, the student was supposed to work through similar exercises almost without any help of written lines of thinking or any other directives. On some few occasions, though, such written lines of thinking were attached. Some of the authors of the textbooks without any written lines of thinking point out that they are applying the so-called *working method*, i.e. *arbetsmetoden* in Swedish.[296]

Introductions to the basic geometrical concepts

A common feature of the textbooks that contained written lines of thinking is that the students are introduced to the basic concepts of point, line, surface, and solid by observing some real object. This was done in two different ways. One of the authors, Lundborg (1918), designed all his explanations in a rather strict format: an object *P* has some properties *A*; that which has the properties *A* is called *Q*; *P* is therefore *Q*. Take for instance the following passage.

> Each side [of a cube] has a length and a breadth. That, which has length and breadth, is called a **surface**; each side of the cube is therefore a surface. … In space, each edge extends in *one direction*, which is in length. That, which has only length, is called **line**; the edges of a die are therefore lines.[297]

In contrast, Knutsson (1922), Danielson (1925), Ohlander & Ingvarsson (1920), Segerstedt (1924) and Dalin (1923) (separated textbooks) along with Nord (1931), Kärrlander (1930) and Rydén (1938) (combined textbooks) employed much more easygoing language. Here is a typical explanation from Knutsson (1922), where the students are supposed to observe a pencil-box and grasp the properties of a line.

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[295] See Lundborg (1918), Danielson (1925), Knutson (1922), Dalin (1923) and Ohlander & Ingvarsson (1920) (separate textbooks) and Rydén, Frank & Norgren (1938) and Kärrlander (1930c) (combined textbooks).

[296] Along with the exercises that are linked to the introductions of concepts and propositions, there are some exercises, but not in all textbooks, where the students are supposed carry out constructions that occur in Euclid's *Elements* Book I. The following constructions occurred: to construct an equilateral triangle, sometimes also an isosceles triangle; to bisect a given rectilinear angle; to bisect a given finite straight line; to draw a straight line at right angles to a given straight line from a given point on it; to describe a square on a given straight line. The construction of a straight line perpendicular to a given infinite straight line from a given point not on it was not treated in any textbook. Of course, the students were also expected to learn how to use a graded ruler, a compass, a protractor or a set-square in order to construct triangles, squares, rectangles, rhomboids and regular polygons.

[297] Lundborg (1918), p. 3: "Varje sida har längd och bredd. Det, som har längd och bredd, kallas en **yta;** var och en av kubens sidor kallas således en yta. ... Varje kant utsträcker sig i rummet i *en riktning*, näml. i längd. Det, som har endast längd, kallas **linje;** tärningens kanter äro således linjer." The bold types are Lundborg's.

> Put a pencil-box in front of you. Take a piece of chalk, and use it to rub the box from one corner to another. The edge that is produced looks like a thin, white stroke or a *line*. Depict it on the blackboard or a paper. Measure the chalked edge using a metric measure. Assume it is 20 cm long. Measure the breadth. No matter how much you try, the measurement will fail, because **lines have length but no breadth**.[298]

The other authors used similar types of explanations where the students are supposed to observe a pencil-box.[299] Here is another typical explanation, this time from Danielson (1925), where the student is supposed to understand the properties of a point.

> Take in each hand a very fine needle. Try to hold the needles in such a way that the tips rest against each other. It fails, because the place where the tip of the needle begins has no extension. It has no length or breadth or height. The place where the tip of the needle begins is a point. There is no outer sign that *fully* corresponds to a point. *A point has no extension in any direction. The point can therefore not be divided and is therefore not a magnitude.[300]*

Some authors of the combined textbooks, Lindström, Jonzon & Jansson (1946), Sandström & Jonsson (1934), Lövgren & Nordström (1926) and Roman & Wigforss (1930), choose a different approach for introducing the basic concepts of point, line, surface, and solid. Their introductions are shorter and the observations of objects are not as experimental as the preceding ones; it is more a question of pointing at objects and giving them names – this is a solid, here are its surfaces and its edges.

> You have perhaps heard the word "water surface", "surface of a wall" and other phrases that end with surface. Give examples of surfaces that you can see a) in the class room b) outdoors![301]

> Some surfaces are curved, others are plane. Give examples.[302]

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[298] Knutsson (1922), p. 3: "Ställ en pennlåda framför dig! Tag en kritbit, och stryk med den från ett hörn på lådan till ett annat! Kanten, som sålunda behandlats, ser ut som ett smalt, vitt streck eller *linje*. Avbilda den på svarta tavlan eller på papperet! Mät med ett metermått den kritade kantlinjen! Vi antaga, att den är 20 cm. lång. Mät bredden! Hur du än försöker, misslyckas denna mätning, ty **linjer hava längd men icke bredd.**" The bold types are Knutsson's.

[299] Danielson (1925), p. 3; Dalin (1923), pp. 3-4

[300] Danielson (1925), pp. 4-5: "Tag i vardera handen en ytterst fin nål! Försök att hålla nålarna så, att deras spetsar vila mot varandra! Det misslyckas, därför att det *ställe*, där nålens spets börjar, icke har någon utsträckning. Det har varken längd, bredd eller höjd. Den plats där nålens spets börjar, är en punkt. Något yttre tecken, som *fullt* motsvarar en punkt finnes icke. *En punkt har icke utsträckning i någon riktning. Punkten kan följaktligen icke delas och är således icke någon storhet.*" Danielson (1925) differs from the others in one interesting aspect; he is the only one that explicitly points out that geometrical objects are not real objects, which we may observe in the quote above. This is done on a couple of other occasions.

[301] Lindström, Jonzon & Jansson (1946), p. 43: "Du har kanske hört orden >>vattenyta>>, >>väggyta>> och andra ord på –yta. Giv exempel på ytor, som du kan se a) inne i skolrummet b) ute!"

Moreover, in these textbooks, the written lines of thinking were dropped, and the students were supposed to work through the exercises on their own. This newer way of introducing the basic geometrical concepts was also applied in the first textbooks for Grundskolan during the 1960's.[303]

Despite these differences regarding the introductions of the concepts of point, line, surface, and solid, all the authors applied very much the same descriptions of the basic geometrical objects and different subcategories of geometrical objects. The students are told that the ends of lines are points, the edges of surfaces are lines, and the sides of solids are surfaces. The students are also introduced to the concepts straight and crooked lines, plane, and bulging surfaces, and they are instructed to identify real objects that have the same properties. They were also told that a straight line is the shortest line between the two endpoints. In several of the textbooks, the students are supposed to measure the lengths of straight and curved threads between two points. However, the Euclidean definitions of a straight line and a plane surface are not used. The authors also introduced the concepts vertical, horizontal, and slanting lines.[304]

Introductions to and explanations of propositions about geometrical objects

Alongside the basic geometrical concepts described above, all textbooks treated the different subcategories of surfaces and solids, such as rectangle, square, triangle, rhomb, rhomboid, circle, ellipse, parallelepiped, cylinder, cone, pyramid, and sphere. In connection to each new subcategory, the students were introduced to how the area or volume is computed. All these introductions followed a certain routine. First, the students work through one or more exercises where they investigate a diagram by means of folding and cutting pieces of paper, by measuring, or by numerical approximations. Eventually, these exercises lead up to a formula or some other proposition.

In this way, an introduction also functioned as some sort of explanation for why a proposition is correct. It was more than simply stating a formula or a proposition and then providing a description of how the formula or the proposition is applied in different settings – this is the area formula for rectangles; you use it like this.

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[302] Lindström, Jonzon & Jansson (1946), p. 43: "En del ytor är buktiga, andra är plana. Giv exempel!"

[303] See for instance Berg & Arvidsson (1964), pp. 58-62; Lindström (1964), pp. 68-69 and Boman & Rydén (1965), pp. 60-63.

[304] Angles were usually introduced by means of an illustration and without any formal definition. Very often, the size of angles is associated with rotation – one straight line is fixed and another straight line is turned around one of its ends. In some cases, rotation is also illustrated by an opening book or door. Angles between curved lines are not mentioned at all. The students also learn how to use a protractor when measuring and constructing angles. In all textbooks, the measure is degrees.

Also at this point, these explanations differed somewhat as some authors submitted written lines of thinking.

In the following sections, I describe how some of these explanations were carried out. To some extent, the explanations were based on the same idea, but there were also apparent differences between the older separate textbooks and the newer combined textbooks.

The area formula for rectangles

In all the investigated textbooks, the introductions to the area formula for rectangles were based on the same idea. Some differed in respect to the written lines of thinking. After being introduced to the area measures of the metric system, the students were supposed to count the square centimetres in a diagram where a number a square centimetres are put in a row. The next step is to do the same thing in a diagram where a greater number of square centimetres are put in two rows.

In some of the textbooks, the students were supposed to measure the top of a book or the desktop by placing and counting pieces of paper having the shape and size of a square decimetre.

The next step was to learn the shortcut where you count the squares in one row and then you count the rows included in the diagram. By multiplication, you then get the number of squares contained in the diagram. The next step was to use a ruler instead of counting squares. The final step was to introduce the A = b · h, but sometimes the formula was left out and introduced only verbally.[305] As you will notice in the quotes below, the term *area* was not used by all authors. Instead they used the terms "surface number", "surface content", "size of a surface" or just "surface".

Lundborg (1918)

> Draw a straight lined surface with 4 equal edges and right angles. This surface is denoted *square*.

> A plane, straight lined, four sided surface, whose sides are equal and the angles are right is called a **square**.

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[305] Kärrlander (1930b), pp. 69-74; Lindström et al (1946), pp. 43-48; Sanström et al (1934a), pp. 79-83; Lövgren et al (1926a), pp. 53-56, 69-71; Roman et al (1929), pp. 48-52; Ohlander et al (1920), pp. 22-23; Lundborg (1918), pp. 12-17; Danielson (1925), pp. 22-26 and Knutsson (1922), pp. 16-19

Note 1. *Surfaces are measured* by other surfaces of a specific size (hectare, are, sqm, etc.). *The square* is considered suitable to measure other surfaces. The size of a surface is called *area* or *surface content*.
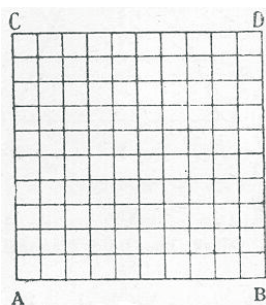
29. Draw a straight line AB, one dm long, and draw a square on this line. *This square is one sqdm*. Divide the line AB in 10 equal parts, likewise the line AC. What is the size of each part? (1 cm) Draw from the intersection points, perpendicular lines to the opposite sides. The small squares that are generated are limited by 1 cm lines, and they are called sqcm. 10 sqcm stand along the line AB and the diagram contains 10 such rows; hence, the whole surface contains 100 sqcm since $10 \times 10 = 100$. From this, we realize that the *surface of a square is attained by multiplying the length number (the base) by the width number (the height)*. If we denote the length number by L, and the width number by W and the surface content by S, then the **formula** for the computation of a square's surface should be this: $L \times W = S$. It is pronounced: the length number times the width number equals the surface content.[306]

Danielson (1925)

The measuring of rectangles

35. Lines are measured by lines of a certain length, e.g. the meter. Measures used to measure lines or lengths are called measures of length, e.g. the meter measure. Angles are measured by angles of a certain size. Measures used to measure angles are called measures of angle, e.g. protractors. Then what is a surface measured by? Surfaces are measured by surfaces of a certain size. What should we then call those measures? Surface measures. Three measures are shown on the next page.

Square millimeter measure, (sqmm)

Square centimeter measure, (sqcm)

Square decimeter measure, (sqdm)

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[306] Lundborg (1918), pp. 12-13. The bold types are Lundborg's

A square-shaped board with sides of one meter is called a square meter measure (sqm.)

Draw a square meter on the blackboard. Practice carefully:

1 sqm. = 100 sqdm … 1/100 sqcm = 1 sqmm

When measuring large surfaces, hectare and are.

1 hectare = 100 are … 1/100 are = 1 sqm.

36. Draw a sqcm on a piece of stiff paper and cut it out? [sic.] What it is this measure called? Cut out 12 such measures. Draw a rectangle, 4 cm long and 3 cm wide. To measure the surface of this rectangle is done in the following way:
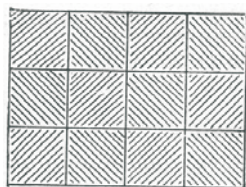
You place the sqcm measures in a row along the length of the rectangle. How many squares fill a row? Now, another row is placed above the first. Above this row, another row is placed. In this manner, the whole rectangle is covered. How many sqcm are needed to accomplish this? Thus, how large is the rectangle? 12 sqcm. To measure *large* surfaces in this way may seem too difficult; indeed, sometimes it is impossible. Therefore, one usually does not *measure* surfaces, one *computes* their size.
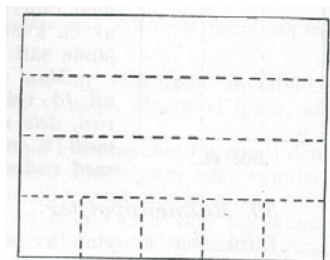
The computation of the rectangle

37. Draw a straight line, 1 cm long. How many sqcm can be placed along this line? Draw a line, 2 cm long. How many sqcm can be placed in a row along that line? Draw a line, 3 cm long. How many sqcm can be placed on it?
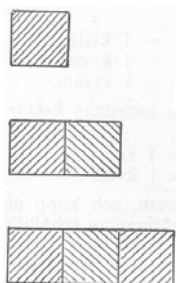
How many sqdm can be placed along a line,

20. 5 dm long?   …   27. 3,4 dm long?

How many sqdm can fill a row along the width of the window? Along its height? How many sqm can be placed in a row along the length of the floor? Along its width? Draw a rectangle, 5 cm long and 4 cm wide. How many sqcm can be placed in a row along the length of the rectangle? How many rows fill the whole surface? Hence, how many sqcm does the whole rectangle contain?


In the computation of the rectangle's surface, you can proceed in the following manner: You measure the length to find the number of surface measures in a row, and you measure the width to find the number of rows. The number of surface measures in a row is then multiplied by the width number of the surface.[307]

Lindström (1946)

584. A square with the side 1 cm is named square-centimetre (abbreviated cm^2).

585. Draw the rectangles *A*, *B* and *C* here below and divide them into to cm^2. Below each surface, write down how many cm^2 it contains.


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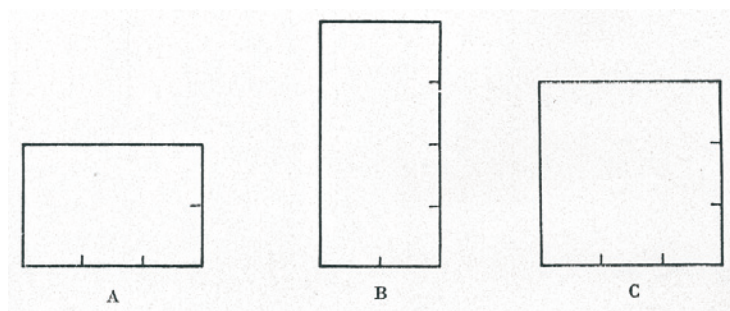
[307] Danielson (1925), pp. 23-25. The italics are Danielson's.

586. Draw other rectangles and compute their size in the same manner: a) 6 cm long, 2 cm wide … j) 5 cm long, 3 cm wide.

587. Instead of drawing cm^2 on a surface, you may measure it by a strip divided into cm^2. See the picture at the top of the page. Make such a strip and measure the rectangles in exercise 585.

588. Draw new rectangles and measure the surfaces by square-centimetres: a) 4 cm long, 5 cm wide … j) 8 cm long, 8 cm wide.

589. Elsa had a small notebook that was 9 cm high and 6 cm wide. Compute the size of each page of the notebook.
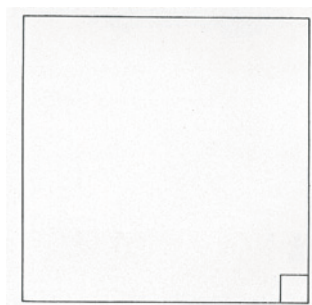
590. A square, whose side is 1 dm, is called a square-decimetre (abbreviated dm^2). How many cm^2 are contained in 1 dm^2.

591. Compute the surfaces of the rectangles that have the following measures: a) 4 dm long, 3 dm wide … h) 9 dm long, 8 dm wide.

592. 1 dm^2 = ? cm^2 … [11 more exercises of this kind]

593. How many dm^2 and m^2 do we get from: a) 120 cm^2 … l) 207 cm^2 ? [308]

Ytan = bredden × längden
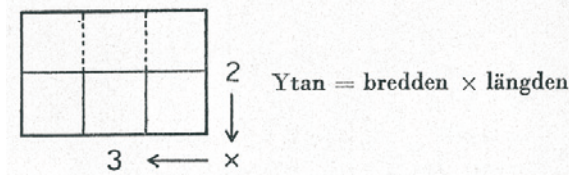
Lindström's approach to the area measure and the area formula was also applied in the first textbooks for Grundskolan in the 1960's.[309]

As the quotes above implies, the authors did not always make a clear distinction between a surface and the measure of a surface, i.e. the area of a surface. A surface is a two dimensional geometrical object, while an area is a

[308] Lindström et al (1947), pp. 44-46
[309] See for instance Berg & Arvidsson (1964), pp. 62-65; Lindström (1964), pp. 70-73 and Boman & Rydén (1965), pp. 66-68.

measure or a number without spatial dimensions. For instance Danielson (1925) uses the expression "The computation of the rectangle"; Lindström (1947) uses the expression "Compute the surface of a rectangle". In the early 1960's, the expression "Compute the surface" occurs in the final exams in mathematics in Realskolan as well. Actually, Lundborg (1918) made a much clearer distinction between geometrical object and measure by stating that "The size of a surface is called *area* or *surface content*." We can observe the confusion of surface and measure in several of the textbooks throughout the period 1905-1962.

In the new curriculum for Grundskolan of 1969, we can observe a difference in this respect. For instance, from the recommendations regarding measurements, it is clear that the authors made a difference between length, area and volume, on one hand, and geometrical objects such as rectangle or solids, on the other hand.[310]

The sum of the angles in a triangle equals 180°

Two "proofs" were quite similar in the sense that the corners of a triangle were placed beside each other. One way is to cut off the corners of a paper triangle and place them on a straight line. Together they constitute 180° or two right angles.[311]

The other way is to fold the paper triangle along the dotted horizontal and vertical lines depicted in the diagram below.[312] The horizontal line bisects the sides of the triangle.
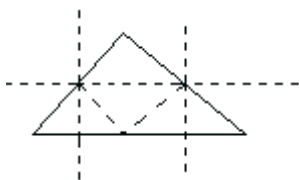
A second approach was to measure the angles and compute their sum. After having done this a number of times, one reaches the conclusion that the angle sum of all triangles equal 180°.[313] This was the approach recommended by Wigforss in his book on methods in mathematics instruction, and it was

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[310] Skolöverstyrelsen (1969), pp. 12-13
[311] Lindström et al (1947), p. 58
[312] Lövgren et al (1926b), pp. 84-85
[313] Kärrlander (1930c), p. 57; Roman et al (1930), p. 62;

applied in his and Roman's textbook. In the textbooks produced for Grundskolan in the 1960's, this was the common approach.[314]

However, the third and most common approach in the investigated textbooks was simply to state the proposition.[315] Thus, the experimental routine was not always applied; there were exceptions.

The definition of π and the formulas for the circumference and area of a circle

The introduction of π was done in the same way in all the textbooks. Having measured the diameter and the circumference of one or more circles, it was described how the ratio between the circumference and the diameter equals almost three. Eventually, the conclusion that the quotient equals 3,14 or 22/7 was reached. In the end, the students were introduced to the equality circumference/diameter = π.[316]

The introductions to the area formula were quite different. There were three approaches, based on three different approximations.

1) In the pre-1930's-textbooks, the authors first derived that the area A of a regular polygon equals the length p of the perimeter multiplied by the length a of the apothem divided by two, i.e. $A = p \cdot a / 2$. In order to derive the area formula for circles, a circle was considered to be a regular polygon with a great number of sides; the radius of the circle was considered to be the apothem of the polygon. Thus, the area A of a circle equals length p of the perimeter multiplied by the length r of the radius divided by two, i.e. $A = p \cdot r / 2$. Since we know that $p = 2 r \pi$, we get $A = \pi r^2$.

2) The students were supposed to compare the surfaces of a circle to the surfaces of two squares, where the sides of the squares equals the radius r and the diameter d of the circle, $d = 2r$.


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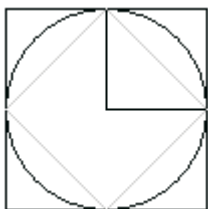
[314] Berg & Arvidsson (1966), p. 130; Lindström (1967), pp. 143-144 and Boman & Rydén (1964b), p. 50. Yet, Berg & Arvidsson (1966) did submit the explanation where a paper triangle is folded.

[315] Sanström et al (1934b), p. 80; Ohlander et al (1920), pp. 41-43; Lundborg (1918), p. 22; Danielson (1925), p. 41 and Knutsson (1922), p. 29

[316] Kärrlander (1930d), p. 70; Lindström et al (1948), p. 58; Sanström et al (1935), p. 88; Lövgren et al (1926c), p. 85; Roman et al (1931), p. 67; Ohlander et al (1920), pp. 13-14; Lundborg (1918), pp. 37-38; Danielson (1925), pp. 49-50 and Knutsson (1922), p. 41

The students were then told that the surface of the circle is greater than the surfaces of two small squares and greater than the surfaces of four small squares, i.e. $2r^2 < A < 4r^2 = d^2$. Without any further comments, the author states that $A = \pi r^2$.

3) A third approach, recommended by Wigforss, was to have the students draw the following diagram on a millimetre-squared sheet of graph paper.
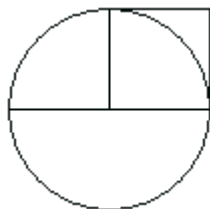
By counting and computing the number of millimetre squares contained by the square on the radius and the quarter of the circle, the students are supposed to calculate the ratio between the area of the circle and the area of the square on the radius. Eventually, this leads to the formula $A = \pi r^2$.

All of these alternatives occurred in the textbooks for Grundskolan.[317] A detail is that the area formula for circles was treated in grade seven in Grundskolan; according to the 1919 curriculum for Folkskolan, this formula should be treated in grade six.[318]

Calculation exercises and applications

In the curriculum of 1919, it was established that the mathematics courses of Folkskolan should provide knowledge suitable for practical daily life. This purpose had great influence on the design of the exercises that were not tied to the introductions of various concepts and propositions. With very few exceptions, the exercises were applied, and the students were supposed to calculate the lengths, areas, or volumes of various objects.

After the initial training in calculation and applying the formulas, the students face some more complicated exercises. However, the increasing difficulty mainly concerns arithmetic in connection to price per unit and different units of lengths, areas, volumes, and currency. Here are some examples of typical exercises that occur in all geometry textbooks of the period in focus.

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[317] Alt. 1) Hultman, Kristiansson & Ljung (1963a), p. 122, Hultman, Hedvall (1963b), p. 194. Alt. 2) Wahlström & Olsén (1963a), p. 54. Alt. 3) Boman & Lindberg (1964c), pp. 52-53; Mattsson, Fredriksson, Göransson & Thulin (1963a), pp. 164-165; Boman & Lindberg (1964c), pp. 52-53; Mattsson, Fredriksson, Göransson & Thulin (1963b), pp. 168-169
[318] Kungl. Skolöverstyrelsen (1919), p. 60

A room is 4 m long and 3 m 5 dm wide: a) How long is the circumference of the floor surface? b) How many lengths of wallpaper are needed for the room? c) What is the cost for the wallpaper if the price is 3 öre per meter? d) How large is the floor? e) What does the varnishing of the floor cost if the price is 40 öre per square meter? f) What does painting the ceiling of the same room cost if the price is 58 öre per square meter?[319]

What is the cost of painting a wall 8 m long and 4 m high, with 3 windows 2 m high and 1 m wide, if the paint costs 1 kr 75 öre per square meter?[320]

a) How much timber does a plank contain if it is 4 m 3 dm long, 2 dm wide and 4 cm thick? b) How much is it for a plank if the price is 20 kr per cubic meter?[321]

What is the weight of a four-sided stone pillar when its height is 15 dm, its width is 8 dm 5 cm and its thickness is 4 dm? We assume that each cubic decimeter of stone is 5 times heavier than 1 cubic decimeter of water that weighs 1 kilogram?[322]

As the concepts rhomboid, triangle, circle, polygon, pyramid, cone, ellipse, and sphere were introduced, the students were supposed to work with similar exercises as the ones above. These types of calculation exercises were the most common in all the investigated textbooks from the period 1905-1962, separated as well as combined. In some of the exercises, the students were supposed to compute the area of the surfaces of a standard solid, e.g. a cube, a cylinder, but also cones.

As a matter of fact, ellipses, cones and spheres were treated in the textbooks up to the 1950's.

However, some of the combined textbooks, which appeared after 1920, contained exercises where the measures of lengths, height, and widths were not directly given and the students could not put numbers into a formula at once; the students had to investigate the geometrical objects along with possible formulas before they could start the calculations. Here are some examples.

How great must the diameter of a circular dining table be if 10 persons are supposed to sit around the table? The distance between each person is 5 dm.[323]

The perimeter of a rectangle equals 8 m 7 dm 6 cm. Its length is a third of the perimeter. What is the rectangle's a) length; b) breadth ; c) surface?[324]

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[319] Danielson (1925), p. 32
[320] Danielson (1925), p. 32
[321] Danielson (1925), p. 31
[322] Knutsson (1922), p. 27
[323] Roman & Wigforss (1931), p. 71
[324] Asperén (1931a), p. 54

The perimeter of a quadratic floor is 26 m. Compute the size of the floor's surface.[325]

The surface of a rectangle is 17,86 m². The width of the rectangle is 3,8 m. What is the size of the perimeter?[326]

Note that in the first example the widths of the persons are not given. Hence, the formulation of the exercise does not provide enough data to solve the problem. Here, the students must estimate the width on their own, unless the persons have zero width.

In all of the separated textbooks, i.e. the textbooks first printed before 1925, there is only one example that is similar to the type of problems listed above. However, in the textbooks printed after 1925, these new types of exercises were not at all as common as the older type of exercises where the students are supposed to decide a formula, plug in the numbers in the right position, and compute an answer.

Another type of exercises that occurred in combined textbooks concerned angles, more specifically, propositions about adjacent angles and the sum of angles in a triangle. Here are some examples.

An angle is a) 67° ; b) 88° ; c) 107° ; d) 146°. How large is the adjacent angle? Draw it.[327]

In a right triangle, one of the acute angles is 38°. What is the size of the other angle?[328]

In a second type of angle exercises, the students are supposed to determine the angles between the hands of a clock. For instance …

What is the size of the angle between the pointers at 5 a clock?[329]

I would say that these clock exercises are a more difficult than the usual computation exercises in the sense that it is not obvious what formulas or propositions you should use. The student must rethink minutes to degrees before they can solve the task. If the clock is not showing whole hours the exercises becomes even more difficult.

Still, in comparison to the number of the older exercises where the students are supposed to calculate areas and volumes, these new angle exercises were very rare.

The separated textbooks did not contain these types of angle exercises even though adjacent angles and the sum of angles in a triangle are men-

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[325] Lindström (1948), p. 71
[326] Lindström (1948), p. 71
[327] Asperén (1931c), p. 84
[328] Sandström (1934b), p. 81
[329] Asperén (1931c), p. 68

tioned. In the separated textbooks, exercises that included angles were measuring and constructions exercises, i.e. the students were supposed to draw and measure angles.

Taken together, all the textbooks during the period 1905-1962, separated as well as combined, contained large sets of computing exercises. The students are given measures of heights, lengths, widths, radius, etc., of some object, often a garden or a building; by putting the numbers in the right formula, the students are supposed to compute areas or volumes. Sometimes the area or the volume was given together with one or two lengths and the students were supposed to compute a length.

In the geometry textbooks for the new Grundskolan, the proportions of the exercises did not change. Also, the routine regarding how to introduce concepts, formulas, and other propositions were kept in these textbooks.[330]

Concluding remarks – the textbooks in Folkskolan and the significance of the professional debate

During the period 1905-1962, the basic elements of the geometry courses in Folkskolan were formulas for computing lengths, areas, and volumes of basic geometrical objects such as rectangles, squares, triangles, parallelograms, circles, and solids made up of such surfaces. A majority of the geometry exercises in the textbooks required the following solution: choose a formula, put in the measurements, and compute the answer. Moreover, most of these exercises had a practical connotation as the students were supposed to compute the length, area, or the volume of various objects; such as a wall or a cylindrical piece of timber. Often, these exercises contained an economic aspect where the students had to compute a price when the price per unit was given. I term this type of exercises the A-type.

After 1925, another type of exercises, requiring more elaborate solutions, were submitted in the textbooks. An example is exercises where the students are supposed to compute the angle between the hands of a clock at a certain time. I call this type of exercises the B-type. The solution to such an exercise was not just a question of choosing a formula, plugging in numbers, and carrying out the calculations. From a mathematical point of view, the B-exercises were a bit more demanding.

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[330] See for instance Boman & Rydén (1965); Boman & Rydén (1964a); Boman & Rydén (1964b), Boman & Lindberg (1964c); Berg & Arvidsson (1964); Berg & Arvidsson (1965); Berg & Arvidsson (1966); Hultman, Kristiansson & Ljung (1963a); Hultman & Hedvall (1963b); Lindström (1964); Lindström (1965); Lindström (1967); Mattsson, Fredriksson, Göransson & Thulin (1963a); Mattsson, Fredriksson, Göransson & Thulin (1963b); Wahlström & Olsén (1963a); Wahlström & Olsén (1962); Wahlström & Olsén (1963b).

In the curricula for Folkskolan during the period 1905-1962, one of the goals of mathematics instruction was to promote knowledge useful in everyday life and working life, together with economic awareness. The A-exercises provide a good picture of how the goal of useful knowledge in mathematics was understood by the textbook authors. Since these types of exercises dominated all the investigated textbooks of the period 1905-1962, it indicates that the goal of providing knowledge useful in every day life and working life was relevant for the textbook authors. Moreover, due this dominance of the A-exercises, I think it is fair to say that goal to promote knowledge useful in everyday life and working life was widely spread among the teachers of Folkskolan.

However, other arguments regarding goals were put forward during this period. Take for instance Wigforss' books on methods for mathematics instruction, published in 1925, where he underscored the value of training in reasoning in connection with studies in mathematics. The occurrence of B-exercises after 1925 may have been an attempt by the textbook authors to meet this goal regarding training in reasoning. But, if B-exercises occurred in the textbooks, they were very few in comparison to the number of A-exercises. Thus, the goal about training in reasoning did not stimulate the textbook authors to include great numbers of the more difficult B-exercises.

The methodological arguments during the period 1905-1962 were centred on the concept visualizability, i.e. *åskådlighet*. The basic argument for arranging mathematics instruction according to the principle was that it counters rote learning and mindless repetitions of rules and formulas. The argumentation about visualizability and mathematics instruction was, however, not just a question of showing an illustration every now and then. A key point was that the students were supposed to observe illustrations or other real objects in an active way. This idea also included that the students should observe, manipulate, or measure an illustration or some other object.

Another key point was that a certain routine should be used when introducing and explaining the meaning of concepts, formulas, and other important propositions. These should not just be listed or stated. Instead, the students were supposed to work through a series of experimental exercises where they observe, measure or manipulates an illustration or some other real object. After these exercises, the definitions, formulas and other propositions were explicitly stated.

In all the textbooks investigated, concepts, formulas, and other important propositions were presented according to this routine, even though there were variations. A clear majority of the authors inserted longer series of experimental exercises, but there are examples where they were very brief. I denote this type of laboratory exercises the C-type.

We might link this change of the C-exercises to the goal about training in reasoning, since the students were supposed to recognize the meaning of a definition, a formula, or some other proposition on their own.

During the period 1905-1962, we can observe a change in the design of the C-exercises. This change occurred in some of the textbooks produced about 1925 and afterwards. Before 1925, the C-exercises were accompanied by short, yet detailed, narratives that guided the students through observations, manipulations, measurements, and the solutions. After 1925, these narratives were left out in some of the textbooks; the students had to tackle the exercises on their own. The illustrations in these later textbooks were also much fewer. However, the routine was still intact where the student works through a series of C-exercises before the definitions, formulas, and other propositions were stated. Actually, none of the textbook authors abandoned this routine during the period 1905-1962.

This change of the C-exercises indicates that the meaning of the concept visualizability changed during the period 1905-1962. After 1925, some textbooks author considered it more important to stimulate the students' ability to observe and reason on their own. Illustrations in the textbooks, on the other hand, were less important.

Taken together, the most significant change of the textbooks during the period 1905-1962 was the modification of the C-exercises and the introductions of concepts and formulas. Thus, I think it fair to say that the methodological directives and arguments about visualizability were relevant to the textbook authors. This idea was something they grappled with as they authored new textbooks.

In contrast, though, there were areas where changes did not take place. As I mentioned above, the A-exercises occurred in the textbooks investigated in great numbers throughout the period 1905-1962, while very few B-exercises were inserted. Furthermore, if we consider the terminology in the textbooks, we can observe that there was some confusion throughout the period 1905-1962. Several textbook authors did not make a clear distinction between geometrical objects and measures, i.e. numbers. For instance, the word 'surface' could sometimes signified a geometrical object, but in the next sentence the student was told to compute a surface, which suggests that surface is a number.

This confusion about the terminology is in some sense linked to the methodological arguments about visualizability. The main aim for the textbook authors was not to develop a coherent terminology by which concepts and formulas could be expressed; their main aim was rather to develop an experimental routine with C-exercises that made the concepts and formulas visual or concrete.

Part E – Geometry textbooks in Realskolan

The supply of textbooks

The geometry textbook offerings in the last decades of the 19th century were quite diversified. If we consider an inventory of textbooks in mathematics at Läroverken, put together by a Government school commission in 1872, the geometry textbooks were designed quite differently. They comprised between 100 and 300 pages. Some of them were more or less modified versions of Euclid's *Elements*. Some of them included trigonometry, but also theories on symmetry, descriptive geometry, and practical applications. To what extent each book was used, at which level they were used, and on which program, the report does not tell. However, most of the textbooks denoted elementary were described as slightly modified versions of Euclid's *Elements*, book I-IV or book I-VI.[331] In the curriculum for Läroverken of 1859, it was established that *Euclid's Elements* should be used as a textbook.[332]

In 1931, another Government school commission completed a survey of the textbooks used in the Swedish schools. According this investigation, the most common geometry textbooks were the following.

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[331] Kommitébetänkande (1872), pp. 109-130
[332] SFS 1859:16, p. 26

Table 6.

	läroverk	mellanskolor	ensk. skolor	sum
Asperén	40	57	34	131
Josephson	13	6	46	65
Vinell	11	12	27	50
Lindman	8	6	6	20
Olson	6	5	3	14
Laurin	-	-	-	10
Others	-	-	-	10
sum	78	86	116	

Numbers of Realskolor at Läroverken, Mellanskolor and Realskolor at Enskilda Läroverk (private schools) that used a particular textbook in geometry in the late 1920's. The numbers are based on statistics in the official report of the Schoolbook Commission of 1927. The report was completed in 1931.[333]

Except for Olson's textbook, the textbooks in the table above had been introduced before 1905 and they were frequently being reprinted.

Table 7.

	Asperén	Josephson	Vinell	Lindman	Olson
1st printing	1896	1900	1898	<1872	1925
last printing	1952	1953	1935	>1927	1954
# printings	21	15	11	>19	10

Of these authors, Vinell and Lindman followed the Euclidean selection of definitions, axioms, and theorems closely, even though they did change some proofs. They also added new definitions, axioms, and propositions. However, they retained the Euclidean outline with just a few alterations. In contrast, Asperén, Josephson, Olson and Laurin gave their textbooks alternative outlines where the theorems where arranged more thematically. However, just like Vinell and Lindman they applied the axiomatic method. Hence, there were no changes in that respect.

Choosing a disposition different from the Euclidean, they were more or less forced to come up with new proofs, even though they did lend some proofs, or parts of proofs, together with constructions from Euclid. The common feature of the alternative textbooks, except for Laurin's, is that they were based on theorems on the properties of the straight lines, perpendiculars, foldings, and symmetry.

At this point we should recall Petrini's defense of Euclid's *Elements* and criticisms of the contemporary geometry textbooks during the 1920's. Con-

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[333] SOU 1931:2, p. 106

sidering the tables above, his criticisms did not concern a recent attempt to develop new textbooks but a widespread use of a certain type textbooks. In this context, it is worth mentioning that Petrini edited Lindman's in the 1920's. Lindman passed away in 1901.

As a backdrop to these tables, we have the continuous expansion of secondary education during the period 1905-1962. Realskolan received an increasing number of students, and new schools were started all over the country. Thus, there was not only an increasing demand for geometry textbooks; at the same time the teaching staffs at the new schools should choose textbooks for the first time. My point is that the introduction of alternatives to Euclid's *Elements* may have been easier because of this. Maybe, old routines and habits were not as dominant in these new schools as in the older schools.

In 1938, the government established a special textbook committee, the so-called *Statens läroboksnämnd*.[334] Its function was to examine the textbooks used in the schools and to regulate the use of textbooks in the schools. The work of the committee resulted in an annual list of approved textbooks. Prior to that, the choice of textbooks was an issue for the teaching staffs alone to decide upon. In January 1941, the committee's first decree came into force. However, the actual list of approved textbooks was completed and published as early as 1935 by a special government commission. In this first list, the traditional Euclidean textbooks by Lindman and Vinell were removed.[335] The textbooks by Asperén, Josephson, and Olson, on the other hand, were being reprinted until the 50's.

After 1925, we can observe two successful attempts to introduce new geometry textbooks on the market. These attempts were made by Olson and Sjöstedt; Olson actually introduced two textbooks, a longer version and a shorter version. They were successful in the sense that their textbooks were reprinted every second or third year. No other textbooks were reprinted with such frequency during the period 1930-1955.

Table 8.

	Sjöstedt	Olson 1	Olson 2
1st printing	1936	1925	1936
last printing	1955	1954	1952
# printings	13	10	8

Olson 1 is the longer version. Olson 2 is the shorter version.

The textbooks by Olson and Sjöstedt were in some respects quite different. Olson pointed out that he was influenced by Meyer's and Asperén's way of

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[334] Statens läroboksnämnd (1940), p. 3
[335] Statens läroboksnämnd (1940), pp. 54-59.

writing textbooks, which deviated from traditional outline of Euclid's *Elements* and the added new theorems that included foldings and symmetry.[336] Sjöstedt, on the other hand, underscored that his intention was to follow Euclid as far as possible.[337] In that sense, Sjöstedt continued the tradition of Lindman, Vinell, and Petrini. However, Sjöstedt also made some modifications which were more extensive than those of Vinell and Lindman.

In 1947, Nyhlén – the critic of Olson's and Sjöstedt's textbooks – published his own textbook in geometry. Even though it was reprinted only twice, it is an interesting textbook since Nyhlén has a much more experimental approach in comparison with the textbooks of Olson and Sjöstedt. Moreover, Nyhlén found that many of the Euclidean propositions and proofs could be omitted or simplified if measures and real numbers were introduced at an early stage.

Another set of new geometry textbooks appeared during the 1950's in connection with the preparations for Grundskolan. The explicit purpose of these textbooks was that they should be used in the experimental schools preceding the introduction of Grundskolan. These textbooks were written by Sandström/Ullemar, Sjöstedt, and Bergström. Sjöstedt's were essentially the same as his previous one, while Sandström/Ullemar and Bergström chose a some what different approach. They put greater emphasis on explaining how a proof was carried out, but they did not deviate from the axiomatic structure. On some occasion, though, they used a theorem before it was proved. In my investigations, I have focused on the textbook by Sjöstedt.

Lindman's and Vinell's editions of Euclid's *Elements*

Lindman's textbook

The textbook by Lindman was first published during the 1860's or the early 1870's.[338] He named the book *The first four books of Euclid*. According to the preface, his motive was to revise Mårten Strömer's edition of Euclid's *Elements*.[339] Strömer's edition was first published in 1744, and it was the first edition of Euclid's *Elements* in Swedish.[340] It was also used as a textbook in

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[336] Olson (1940), pp. v-vi
[337] Sjöstedt (1936), pp. 5-6
[338] I have not found a date on the first printing, but second took place in 1872.
[339] Lindman (1897), p. iv
[340] Rodhe (2002), p. 7. The same year, the Swedish edition of Clairaut's introduction to Geometry was published. Clairaut's textbook is very different from Euclid's *Elements*; it contains no proofs according to the axiomatic method, and it includes real numbers and examples of practical applications.

the secondary schools throughout the 18th and 19th centuries.[341] The last edition of Strömer's book was published in 1884.[342]

In a comparison with Heath's edition of Euclid's *Elements*, Lindman included all propositions of Euclid's Book I and III, except for Prop.I.7. Lindman also retained the order of the propositions. The omission of Prop.I.7 is quite easy to overcome in the sense that Prop.I.7 is used only in the proof of Prop.I.8, i.e. the SSS congruence theorem. Hence, this change did not force Lindman to revise the proofs of the later propositions.

Lindman motivated the cancellation of Prop.I.7 by saying that students had difficulty understanding it.[343] Lindman proved Prop.I.8 in the following way.

> Prop. VIII If two sides *AB* and *AC* in a triangle *ABC* are equal to the sides *DE* and *DF* in another triangle *DEF* and the base *BC* in the former is equal to the base *EF* in the later, then the triangles are congruent.
>
> Hypotheses: *AB=DE, AC=DF, BC=EF*.
>
> Thesis: $\triangle ABC \cong \triangle DEF$
>
> Move $\triangle DEF$ and place point *E* at *B* and *EF* along *BC*, then point *F* falls in *C*, since *BC=EF* (hyp.); give to $\triangle DEF$ a position such as $\triangle GBC$. Since $\triangle EBC$ [sic] is the same as $\triangle DEF$, in another position, then *CG=DF=AC*, *BG=DE=AB*, $\wedge BGC = \wedge D$. Connect *AG*, which then will fall either a) on the same side of *B* and *C* or b) go through one of them, e.g. *C* or c) fall between *B* and *C*.

Fig. 22 a).   Fig. 22 b).   Fig. 22 c).
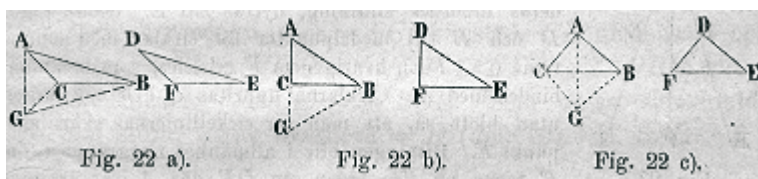
> a) Since *AB* = *BG*, $\wedge BAG = \wedge BGA$ (prop.5), and since *AC* = *CG*, $\wedge CAG = \wedge CGA$. If they are removed from the former, then (ax.3) $\wedge BAC = \wedge BGC = \wedge D$. Since *AB* = *DE*, *AC* = *DF* and $\wedge BAC = \wedge D$, then $\triangle ABC \cong \triangle DEF$ (prop.4).
>
> b) In this case it immediately follows by prop. 5, that $\wedge A = \wedge G = \wedge D$, therefore the triangles are congruent.

c) Since $AB = BG$, then $\angle BAG = \angle BGA$ (prop. 5) and since $AC = CG$, then $\angle CAG = \angle CGA$. If the latter is added to the former, then (ax. 2) $\angle BAC = \angle BGC = \angle D$ and then the triangles are congruent (prop. 4). Q.E.D [344]

Lindman also shortened some proofs by several steps by applying movements of triangles. A good example of that is Lindman's proof of Prop.I.5.

Prop. V The angles B and C, which stand against the equal sides of an isosceles triangle ABC, are equal.

Hypothesis: $AB=AC$;

Thesis: $\angle B=\angle C$

Let $\varDelta ABC$ be moved so that it gets the position $A'B'C'$ where $\angle C'=\angle C$, $\angle B'=\angle B$. Then $\varDelta A'B'C'$ is in all respects similar to $\varDelta ABC$, except for its aforementioned position. Now you can show, in the same manner as in prop.4[345], that $\varDelta A'B'C' \cong \varDelta ABC$, thus $\angle C'=\angle B$; but $\angle C'=\angle C$, hence $\angle B=\angle C$ (ax. 1)[346]. Q.E.D.[347]
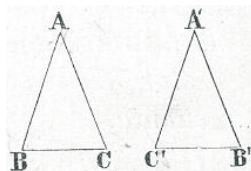
We can compare this to the construction and the proof in Heath's editions of Euclid's *Elements*. (See Appendix A)

A detail in Lindman's two proofs above is that the diagrams explain how the triangles are moved. The text does not reveal how the triangles are placed and without the diagram the proof is difficult to follow. Actually, when Lindman described how the triangle is moved, in writing, he did not change the order of the letters. He could he had written that $\varDelta ABC$ is moved so that it gets the position $A'C'B'$ in order to clarify the change in position of the base angles. This reliance on the diagram was quite common in the textbooks investigated. We can observe it in the textbook by Vinell as well in the alternative textbooks by Asperén and Olson. Quite often they did not explain in writing how the diagrams were moved. A textbook author who deviated from this praxis was Sjöstedt. In his textbook, first printed in 1936,

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[344] Lindman (1897), pp. 16-17

[345] Prop.4 is the SAS congruence theorem, i.e. Prop.I.4 in Euclid's *Elements*, Heath's edition. The procedure Lindman was referring to is applied in Euclid's *Elements*, Heath's edition. [Heath (1956), vol. 1, p. 247-248]

[346] Axiom 1: "They which are equal to one and the same are mutually equal." [Lindman (1894, p. 10]

[347] Lindman (1897), pp. 15-16.

the movements of diagrams were specified in words. I exemplify this in the chapter on Sjöstedt's textbook.

Another aspect of Lindman's proof of Prop. V is that it is hard to follow how he reached the conclusion that the base angles are equal. He explained how the triangle was moved a first time, but then he left out the part were the triangles are placed on each other; he merely referred to the proof of the previous theorem, the SAS congruence theorem.[348] By placing the triangles on each other, he obtained that $\Delta\ A'B'C' \cong \Delta\ ABC$, which entails that $\wedge C' = \wedge B$. However, it is not the establishment of congruence that is the critical feature of the proof. The important premise is that $\Delta\ ABC$ is an isosceles triangle, i.e. $AB = AC$, but Lindman's proof does not draw attention to this circumstance nor does it show how the premise is used.

Moreover, since we are dealing with same triangle, but in another position, the conclusion that $\Delta\ A'B'C' \cong \Delta\ ABC$ may appear strange. Especially, when Lindman did not change the order of the letters, he wrote $\Delta\ A'B'C' \cong \Delta\ ABC$. A change of letters to $\Delta\ A'C'B' \cong \Delta\ ABC$ can make the new position more apparent.

We can compare Lindman's proof to Sjöstedt's, where triangles are placed on each other and the premise $AB = AC$ is used explicitly.

Given: In ΔABC, $AB = AC$.

Claim: $\wedge B = \wedge C$.

Proof: The triangle is lifted up from its plane, turned around and replaced so that $\wedge A$ comes in its former position, but its legs have changed position with each other.
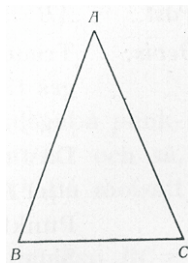
Then point B falls in the former position of point C and point C falls in the former position of point B, since $AB = AC$ (given).

Side BC then falls on CB's former position. (ax.1)[349]

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[348] In the proof of Prop.IV, Lindman applied the same procedure of superposition that we find in Euclid's *Elements*, Prop.I.4, Heath's edition. [Heath (1956), pp. 247-248]

[349] Sjöstedt's Axiom 1: "Through two points, only one straight line can run." [Sjöstedt (1936), p. 12]

⌒B then covers the former position of ⌒C.

∴ ⌒B=⌒C (ax.11)[350]

Q.E.D.[351]

If we return to Lindman's proof of Prop. V, it is especially interesting in relation to Petrini's appeal about improving the level of precision in the geometry textbooks; but also in relation to his argument that a high level of precision made the proofs easier for the beginner. When Lindman's textbook was republished in 1922, after Lindman had passed away, Petrini had re-edited some parts. He did change the order of the letters in the proof of Prop.V, he wrote $\Delta\ A'C'B' \cong \Delta\ ABC$, which can make the proof easier to follow. However, he did not describe the second part of the proof when the triangles were placed on each other.[352] Hence, Petrini did not draw attention to the important premise $AB=AC$ and how it was applied.

Asperén, on the other hand, inserted another version of this proof where he describe the whole process of placing one of the triangles on the other, a proof quite similar to Sjöstedt's.[353]

Let us return to Lindman's textbook once again. Lindman not only moved triangles more often than Euclid, he also moved circles. In this way, Prop.I.2 also got a shorter proof.

> To draw a straight line from a given point A that is equal to a given straight line BC.
>
> Draw from A, at will, a straight line AG and place thereafter from A the line BC (Post. 2; Ax. 8). This is done by means of the tool mentioned in Def. 16, that is called circle or compass, whose legs are put at B and C. When one leg is moved to A, then the other determines the sought point G [sic] on AG. This is apparently the same thing as drawing a circle with the centre at A and the radii =BC (Post. 3).[354]

However, Lindman did include the Euclidean proof of Prop.I.2 as well.

Even though Lindman cared for the Euclidean order of the theorems, this did not prevent him from adding new theorems. These were inserted as soon as the required theorems for the proofs had been established. In order to retain the Euclidean numbering of the theorems, the new theorems were given

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[350] Sjöstedt's Axiom 11: "Quantities that can cover each other are equal in size" [Sjöstedt (1936), p. 14]
[351] Sjöstedt (1936), pp. 21-22. According to Heath (1956), this proof was by no means an innovation of the late 19th century. Heath (1956) links it to a proof of Pappus, 300 AD. [Heath (1956), vol. 1, pp. 254-255]
[352] Lindman (1922), p. 27
[353] Asperén (1928), p. 36; Sjöstedt (1936), pp. 21-22
[354] Lindman (1897), p. 13

an index letter. Take for instance Prop.XXXIV.B that was inserted between Prop.XXXIV and Prop.XXXV.

> The straight line that joins the middle point of the hypotenuse in a right an-gled triangle and the point of the right angle equals half the hypotenuse.[355]

Regarding the definitions, postulates, and axioms, they were almost the same as those we find in Heath's edition of the *Elements*. Lindman added a few new definitions, and the fourth and the fifth postulate were put among the axioms. He also added two axioms about magnitudes and inequalities and two axioms about magnitudes, equalities, multiples, and quotes.[356] A fifth new axiom concerned the properties of straight lines.

> Two straight lines cannot enclose any space [sic], thus they do not form a figure.

> Thereof follows that two straight lines, which have two points in common, coincide.[357]

Vinell's textbook

If we compare Lindman's and Vinell's textbooks, Vinell used the same defi-nitions, postulates, and axioms as Lindman. He too named his textbook *The first four books of Euclid*. One difference, though, was that Vinell attached visual explanations to the definitions.

> 2. 3. A stroke, drawn along a ruler, is in common language called a *line* or more carefully a straight line. The finer the stroke is made, the more it be-comes a **mathematical line**, which however does not have any width or thickness whatsoever, but length only; it can therefore merely be perceived by thought.[358]

Vinell, too, retained the Euclidean disposition of the propositions, but Prop.I.7 was excluded and Prop.I.5-6 was moved to follow upon Prop.I.17 and Prop.I.18 respectively. Moreover, Prop.III.13 was inserted between Prop.III.10 and 11, and Prop.III.23 was excluded. Even though these propo-sitions had been relocated, they were still given the original Euclidean num-bering.

   Also in Vinell's textbook, we can observe attempts to make the proofs shorter and less complicated. In the proof of Prop.I.2, he applied rigid movements of circles in the same way Lindman did. In contrast, though, Vinell excluded the original Euclidean proof.

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[355] Lindman (1897), p. 39
[356] Ge exempel på dessa.
[357] Lindman (1897), p. 11
[358] Vinell (1898), p. 9. The bold types are Vinell's.

Vinell also ran into difficulties when he modified the proofs. Since Prop.I.5 was relocated, it appeared after Prop.I.16. He had to accomplish a new proof for Prop.I.8, i.e. the SSS congruence theorem.

> Prop. 8 If all three sides in a triangle are equal to each of the sides of another triangle, then the triangles are congruent.
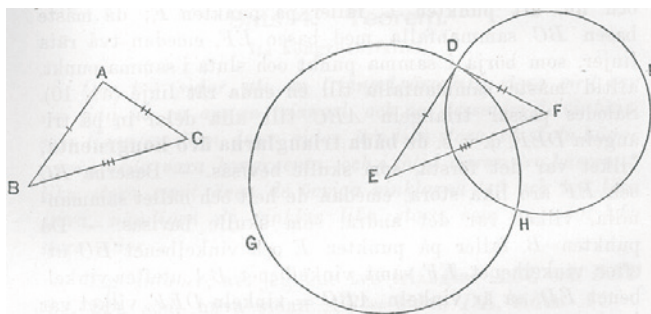
> I assume that *ABC* and *DEF* are two triangles that have *AB=DE*, *AC=DF* and *BC=EF*, and I claim that *ΔABC* ≅ *ΔDEF*. For the sake of the *proof* I begin to make a *construction*. I take *E* as a center and I draw a circle, whose periphery also runs through *D*, and I take *F* as a center to another circle whose periphery also runs through *D*, a point that becomes one of two intersections of the two peripheries; the other, *H*, always falls on the other side of *EF* (compare def. 18, comment. 3). Thereafter I imagine that I place *ΔABC* on *ΔDEF* in such a way that the point *B* falls on the point *E* and the base *BC* falls along the base *EF*; then the point *C* must fall on the point *F*, since *BC* is assumed = *EF*. When now the point *B* falls on the center *E*, then the point *A* falls somewhere on the periphery *GDH*, since *BA* is assumed to = the radii *ED*; but when the point *C* falls on the center *F*, then the point *A* must also fall on the periphery *IDH*, since *CA* is assumed to = radii *FD*. Thus, when the point *A* shall be lie on both peripheries *GDH* and *IDH*, then it must lie on their common point *D*, but then also the side *BA* must coincide with the side *ED*, since they are straight lines, that begin in the same point and end in the same point, and for the same reason also the side *CA* must coincide with the side *FD*. **Therefore, both triangles meet each other parts or they are congruent.**[359]

An important part of this proof is the fact that the two circles intersect in two and only two points; otherwise it is not certain that the sides of the two triangles coincide. Vinell was here referring to a comment to the definition of a circle. In the comment he established that two circles, which do not touch and do not coincide, have two intersections.[360] However, he did not say that there are exactly two intersections. Even more importantly, the proposition on two circles having at most two intersections is a theorem in Euclid's *Ele-*

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[359] Vinell (1898), pp. 24-25. The italics and bold types are Vinell's. In this proof, Vinell is much more careful in his writing than in the later proofs. In the later proofs, he is not as explicit about why he makes certain constructions.
[360] Vinell (1898), p. 11

ments, i.e. Prop.III.10. Thus, Vinell's proof of Prop. 8 is not really a proof since he used a theorem not yet proven. However, the status of the proposition is ambiguous since he did not attach a proof to Prop.III.10.

The alternative textbooks of Asperén and Olson

There are a lot of differences between Asperén's and Olson's textbooks, on one hand, and Lindman's and Vinell's, on the other. In the subsequent sub-chapters, I give some examples of such differences, but also some similarities.

An outline of Asperén's textbook

Compared to how Lindman and Vinell kept almost every definition, axiom and proposition of the *Elements* along with the disposition of the theorems, Asperén made a clear break with this tradition on several points.

1) The basic concepts were introduced in a preliminary chapter on visualizations called *Åskådningslära*. In this chapter the students were supposed work through series of exercises and become aware of geometrical objects and the terms used to differentiate between them. The exercises in this chapter were similar to the exercises in the textbooks at Folkskolan where the students were supposed to observe or manipulate real objects or illustrations.

2) There were no explicit lists of definitions. But Asperén did include an intial list of axioms, but these concerned quantities in general, not lines and surfaces in particular. The term postulate was not used.

3) He did, however, include some "self-evident propositions" or "axioms" regarding straight lines; these were submitted as the first seven propositions, but without proofs. The propositions were:

Prob.1   To draw a straight line through a given point.

Prob.2   To draw a straight line through two given points.

Th.I.a   Through two points, only **one** straight line can be drawn.

Cor.1   Two straight lines cannot intersect each other in more than *one* point.

Cor.2   If two straight lines have two points in common, they coincide.

Th.I.b   The straight line is the shortest distance between two points.

Prob.3   To construct a) the sum b) the difference of two given straight lines.

Th.II.a If the legs of an angel are placed in a straight line (in different directions), then the angle is 2R.

Th.II.b If the angle =2R, the legs of the angle lies in a straight line.[361]

4) The outline of the book was thematic and divided into the following themes: "Self-evident proposition (axioms)" -- "Straight lines, angles and triangles" -- "Parallel lines and parallelograms" -- "Geometrical positions" -- "The circle" -- "Diagrams with equal surfaces" -- "Regular polygons" -- "Uniform mapping".

   5) Asperén added some new propositions and the Euclidean order of the propositions was set aside. In Appendix D, I have inserted a list of the theorems in Asperén's first chapter "Straight lines, angles, and triangles". Here, you can see which Euclidean propositions that he kept. Moreover, different sets of theorems are not dependent of each other. Hence, the sets of theorems IV.a-d, V.a-c, VIII.a-d and IX.a-X.b do not necessarily come in this order.

Asperén's theorems on straight lines, perpendiculars and foldings

The initial theorems III.a-b and IV.a-d concerned straight lines and perpendiculars. Theorem III.a was important since, together with foldings, it served as a substitute for the SAS congruence theorem in some proofs. This theorem is in some sense a special case of the SAS congruence theorem where the adjacent angles equal a right angle. The difference is that the triangles are reversed.

   III.a) If a straight line is perpendicular to another [straight line] and the later is bisected by the perpendicular, then each point on the perpendicular is placed on equal distances from the endpoints.
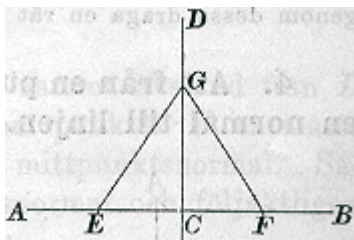
   Given: $CD \perp AB$; $CE=CF$; G is an arbitrary point on CD.

   Claim: $GE=GF$.

Proof: If the figure is folded along *CD*, then *CE* will fall on *CF* as, ∧*GCE*=∧*GCF* , and *E* will fall on *F*, as *CE=CF*. Since *E* falls on *F* and the point *G* remains in its position, it follows that *GE=GF*.[362]

This theorem was often applied in the subsequent theorems. A peculiar circumstance is that the exact formulation in the theorem was often not used in the proofs. Instead the following proposition was used implicitly:

If a point A lies on the same distance from the points B and C, then A lies on the midpoint perpendicular to the straight line BC.[363]

This is a consequence of the last part of the Theorem III.a: "each point on the perpendicular is placed at equal distances from the endpoints." However, Asperén did not discuss this aspect of Theorem IIIa. Neither did he give explicit references to the theorem in the proofs. Take for instance the problem of bisecting an angle. Here, Theorem IIIa was used in the second sentence of the proof.

7. To bisect an angle.

Given: ∧*A*.

Sought: The straight line that bisects ∧*A*.

Solution: Take *A* as the centre to a circle with an arbitrary radii that intersects the legs of the angle in *B* and *C*. Then take *B* as a centre to a circle, whose radii is > half the distance between *B* and *C*, and take *C* as a centre to a circle with the same radii; these circle lines intersect each other in *D*. Draw *AD*.

Claim: *AD* bisects ∧*A*.

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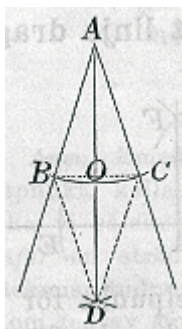
[362] Asperén (1939), p. 27. Th.III.b) was formulated in the following was. "Each point not on the perpendicular is closer to the endpoint at the same side [of the perpendicular]."
[363] This consequence was formulated and applied by Olson in his textbook, which I show in the next section.

Proof: Connect B and C. BC intersects AD at O. According to the construction, A is located at equal distances from B and C; the same is valid for the point D, then AD is a midpoint perpendicular to BC. If the diagram is folded along AD, then OC falls on OB, since $\angle AOC = \angle AOB$, and C coincides with B, since $OC = OB$. Since A remains in its position and C coincided with B, then AC coincides with AB and thus the angles at A will cover each other and they are therefore equal.[364]

Moreover, in the proof, the inference in the second sentence is not correct. The fact that A and D are located at equal distances from B and C means that A and D lie on the midpoint perpendicular to the line BC. What Asperén did not explain and justify is that the midpoint perpendicular to BC coincides with the line AD.

Theorem III.a was also applied in theorems on circles, for instance in the last part of the proof to Theorem XXXVIII.

XXXVIII. Tangents that fall on a circle from a point outside a circle are equal; the straight line from the centre to the point outside the circle bisects the angle between the tangents and it is the midpoint perpendicular to the straight line between the points of contact.
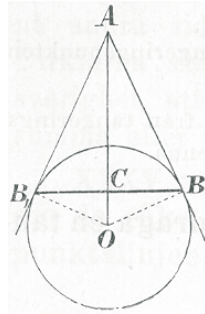
Given: AB and AB_1 are tangents to the circle O in B and B_1.

Claim: 1) $AB = AB_1$

         2) $\angle BAO = \angle B_1AO$

         3) $BC = B_1C$, $AC \perp BB_1$.

Proof: $\triangle BAO \cong \triangle B_1AO$ (SSA congruence theorem[365])

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[364] Asperén (1939), p. 30
[365] Asperén (1928), p. 42: "If in a triangle two sides and one opposite angle equal the corresponding elements in another [triangle] and in addition the other opposite angles are of the same kind [obtuse, right or acute], then the triangles are congruent."   ~"Om två sidor och en motstående vinkel i en triangel äro lika med motsvarande element i en annan och därjämte de övriga motstående vinklarna är av samma slag, så äro trianglarna kongruenta."

$$\therefore AB = AB_1 \text{ and } \wedge BAO = \wedge B_1AO.$$

Since $AB = AB_1$ and $OB = OB_1$, AO is the midpoint perpendicular to BB_1.[366]

Neither in this proof did Asperén explain that AO and the midpoint perpendicular to BB_1 coincide.

Since Asperén from time to time did not care to make explicit references to Theorem IIIa, or to explain the consequence of this theorem he was using, or to explain and justify that two lines coincide, it appears as if he considered these parts of the proofs to be self-evident. At this point, we should recall Olson's arguments about leaving out axioms and self-evident propositions in order to not confuse the students with abstract formulations. We should also keep in mind Meyer's proposal to use proofs that revealed the "nature" in the theorems. Meyer argued that proofs that were based on foldings or symmetry were more in tune with the "real grounds" or the "nature" of the theorems.[367] Perhaps, Asperén had something similar in mind as the authored his textbooks.

An important aspect of having a substitute for the congruence theorems is the importance of the latter in Euclid's *Elements*. To Sjöstedt, for instance, this was very important.

> The most distinctive feature of this course [the Euclidean] can be said to be the central position of the theory of congruency. [368]

Moreover, if we also consider the final exams in mathematics in Realskolan, see Part E of this dissertation, the congruence theorems were useful in the solutions to several of the geometrical problems. Thus, when Asperén inserted these theorems, they were not an innovative detail that facilitated a reorganization of the theorems. He truly got to the core of the geometry courses.

Olson's theorems on straight lines, perpendiculars, foldings and symmetry

Olson used the same basic propositions on straight lines and perpendiculars as Asperén did. They were denoted Prop. 4a and b, and they were proved in the same way. However, Olson added a comment to these theorems, the comment that Asperén did not provide.

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[366] Asperén (1939), pp. 70-71. This theorem is not included in Euclid's *Elements*, but we find in both Olson's and Sjöstedt's textbooks.
[367] See Part C for the arguments of Meyer and Olson.
[368] Sjöstedt (1936), p. 7

From proposition 4 (a and b) follows that each point that lies at equal distances from the endpoints of a straight line must lie on the midpoint perpendicular of the straight line.[369]

He also made explicit references to this comment in the subsequent proofs; see the proof of Olson's Prop.30 below. In this respect, Olson was more formal than Asperén. Here we can notice that Olson's textbook was first printed in 1925. Some years earlier, in 1918, Petrini had published an article where he urged textbooks author to make the textbooks more exact.[370] So even though Olson did not share Petrini's view on elementary geometry instruction in general, Olson did not ignore arguments about the need for more exact textbooks.

Let us return to Olson's textbook again. In connection to Prop.8, Olson introduced the *symmetry* concept, a concept he explains and defines by means of foldings and congruence.

> If, *without* conversion, you move two convertible congruent figures in the plane, so that two corresponding sides
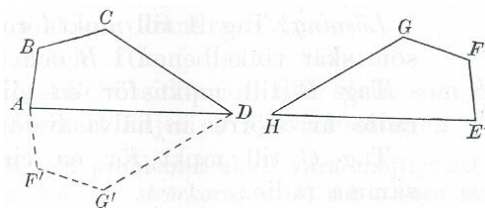
> coincide and the figures are turned in the same direction, then they together form one figure that has the common side as a *line of symmetry* or an axis of symmetry. A symmetry line to a figure is a line so constituted that if one part of the figure is folded about this line, then the two parts of the figure will coincide completely. If for example the figure *EFGH* above is turned around in the plane in such a manner that *EH* coincides with *AD*, then it attains the position *AF′G′D′*, and the figure *ABCDG′F′* is formed, which has *AD* as the symmetry line.[371]

Considering Meyer's argument that proofs that were based on foldings or symmetry should be more natural, we can observe that Olson explains the meaning of a symmetry line by stating that two figures are congruent. Thus, he is appealing to the reader's understanding of congruence in order to explain symmetry. Following Meyer's arguments, it should perhaps be the other way around. However, Olson did explain congruence by referring to foldings.

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[369] Olson (1940), p. 11: "Av sats 4 (a och b) följer, att varje punkt, som ligger lika långt från en sträckas ändpunkter, måste ligga på sträckans mittpunktsnormal."

[370] See Part C of this dissertation for Petrini's arguments.

[371] Olson (1940), p. 15

The symmetry concept was then inserted in Prop. 8.

Prop.8 If an arbitrary point on the midpoint perpendicular to a straight line is joined with the endpoints of the straight line, a triangle is generated that has the midpoint perpendicular as a symmetry line.
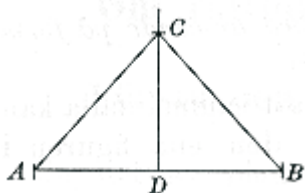
Assumed: *C* is an arbitrary point on *CD*, which is the midpoint perpendicular to *AB*.

Claim: *CD* is the symmetry line to *ΔABC*.

The proposition is proved in 4a.

The parts of the triangle are congruent. In particular we point out that both angles at *C* are equal; moreover ∧*A* = ∧*B*.[372]

The proposition is proved in 4a since it is shown in the proof that *ΔADC* coincides with *ΔBDC* as *ΔADC* is folded around *DC* (See Asperéns proof of Th.III.a above). According to the definition of a symmetry line, *DC* is then a symmetry line.

Olson used this Prop. 4 and 8 in various proofs. In comparison to Asperén, he extended the applications of the theorems on foldings and symmetry. One example is the proposition on base angles in isosceles triangles.

Prop.30. If two sides in a triangle are equal, then the opposite angles are equal.
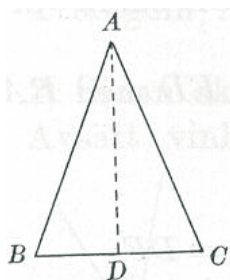
Assumed: In *ΔABC*, *AB=AC*.

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[372] Olson (1940), p. 16

Claim: $\wedge B = \wedge C$.

Proof construction: Join A and the midpoint D of the side BC.

Proof: A lies on the same distance from B as from C (assumed); thus A lies on the midpoint perpendicular of BC. (Prop.4) [Here is Olson using the comment on Prop.4 that I mention above.]

So does D. (Construction)

Therefore, AD is the midpoint perpendicular to BC. (Prop.4).

$\therefore$ AD is the symmetry line to $\triangle ABC$ (Prop.8)

$\therefore$ $\wedge B = \wedge C$.[373]

I think this proof is a good example of how Olson replaced the Euclidean congruence theorems by his Propositions 4 and 8. Prop.30 could easily have been proved by means of the SSS congruence theorem. After having constructed the midpoint D and drawn the straight line AD, we have two triangles *ABD* and *ACD* with mutually equal sides. By the SSS congruence theorem, which was Prop.11 in Olson's textbook, we know that the base angles are equal. I am not fully convinced that Olson's proof was easier for the students to follow.

In some sense, Olson must have considered Props. 4 and 8 more fundamental than the Euclidean congruence theorems. Also in the proof of the congruence theorems SSS and SSA, Prop.4 and Prop.8 were applied. However, the SAS congruence theorem was proved by means of superposition in same manner as in Euclid.[374] Here is the proof of the SSS-congruence theorem.

Prop.11. Theorem. If the three sides of a triangle are equal to the sides of another triangle, the triangles are congruent.
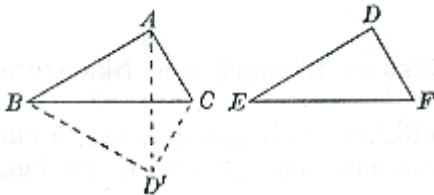
Assumption: In *ΔABC* and *ΔDEF*, *AB=DE*, *AC=DF*, *BC=EF*.

[373] Olson (1940), p. 36
[374] Olson (1940), p. 20; Heath (1956), vol. 1, pp. 247-248.

Claim: $\Delta ABC \cong \Delta DEF$.

Proof construction: Move ΔDEF so that the side EF coincides with the side BC and put ΔDEF in such a manner that the point D gets the position D'. Draw AD'.

Proof: $AC=DF$ (Assumption)

But $D'C$ is the same line as DF.

$\therefore AC=D'C$.

Moreover, we got $AB=DE$ (assumption)

But $D'B$ is the same line as DE.

$\therefore AB=D'B$.

Thus, C lies on the same distance from A as from D', i.e. C lies on the midpoint perpendicular to AD'. (Prop.4)

The same goes for the point B. [♥]

From this follows, BC is the midpoint perpendicular to AD' (Prop.4, comment).

$\therefore BC$ is the symmetry line to $\Delta AD'B$ as well as $\Delta AD'C$ (Prop.8).

From this follows immediately, $\Delta ABC \cong \Delta D'BC$.

But $\Delta D'BC$ is ΔDEF in another position.

$\therefore \Delta ABC \cong \Delta DEF$ (Ax.9)[375]

A detail in the proof is that Olson relies on the illustration as he described the position to which the triangle EDF is moved. Without the diagram, you have trouble following the constructions and the proof.

Another detail is that every inference in the proof was not justified correctly. The conclusion that BC is the midpoint perpendicular to AD' is based on a comment to Prop. 4.

Since a straight line is fully determined by two points (page. 4), you then need to seek only two points in order to construct the midpoint perpendicular

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[375] Olson (1940), p. 18. Olson's Axiom 9: "Hereby we assume as self-evident that lines and figures that we imagine to change position in space *thereby do not undergo any kind of change with respect to shape and size*" [Olson (1940), p. 15]

of a straight line, points that lie at equal distances from the endpoints of the straight line, and then draw the line that is determined by these two points.[376]

However, from this comment it does not follow that the midpoint perpendicular to AD´ coincides with the original line BC. Because, when Olson got to ♥, he had thus far proven that the points B and C lies on the midpoint perpendicular to AD´, not that the straight line BC is the midpoint perpendicular to AD´.
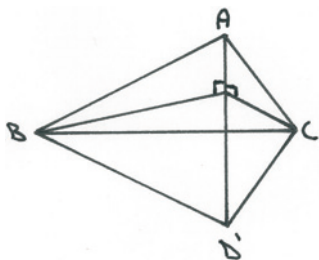
In order to do so, he should have had an axiom that established the property that through two points, one and only one straight line can run. Olson did mention this property in a discussion on the properties of points and straight lines, but he did not formulate an explicit axiom.[377] Hence, Olson did not enhance all parts of Asperén's textbook with respect to rigor and exactness. Moreover, I think that the proof of Prop.11 is a good example of what Olson meant by putting pedagogical requirements before scientific. I am now referring to Olson's argument about avoiding axioms and self-evident propositions that may confuse the students.[378]

Asperén and Olson and movements

In comparison to Lindman and Vinell, both Asperén and Olson applied new types of movements. In the proof of the propositions on straight lines and perpendiculars, we can observe that both Asperén and Olson applied foldings. They also applied movements when figures were lifted and turned around. Indeed, in the debate in the journal *Elementa*, Olson argued that movements should be used generously in connection with elementary geometry instruction.[379]

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[376] Olson (1940), p. 11: " Emedan en rät linje är fullt bestämd av två punkter (sid. 4), behöver man för att konstruera en sträckas mittpunktsnormal blott söka två punker, som var för sig ligga lika långt från sträckans ändpunkter, och sedan draga den av dessa två punkter bestämda räta linjen."
[377] Olson (1940), p. 11
[378] See Part C for Olson's arguments.
[379] See Part C for Olson's arguments.

In the proof on parallel lines and alternate angles, Prop.I.27 in the *Elements*, Asperén and Olson introduced movements of open figures. Recall that Lindman and Vinell moved only closed figures, such as triangles and circles.

Prop.17. Theorem. If two straight lines are intersected by a third and a pair of alternate angles is equal, then the first two lines are parallel.
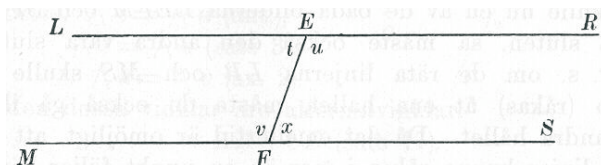
Assumption: The lines *LR* and *LS* are intersected by the line *EF*, and the alternate angles *u* and *v* are equal.

Claim: *LR // MS*

Proof: $\wedge u = \wedge v$ (assumption)

$\wedge t$ is a supplement angle to $\wedge u$ (prop 2)

$\wedge x$ is a supplement angle to $\wedge v$ (prop 2)

$\therefore$ $\wedge t = \wedge x$
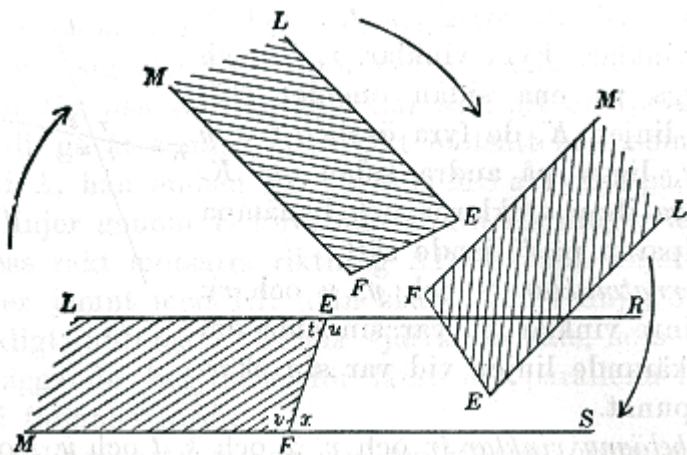
If one considers the diagram cut along *EF* and then turns the left (patterned) picture *LEFM* around in the direction of the arrows, as the sketch suggest, then one gets it to coincide completely with the right (unpatterned) picture *SFER*. Because, if the left picture is turned in such a manner that point *F* in that picture falls in *E* in the right picture and *E* in the left picture falls in *F* in

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the right picture, then the line *FM* falls along *ER*, since $\wedge v = \wedge u$ (assumption) and the line *EL* along *FS*, since $\wedge t = \wedge x$ (recently proven)

> Should then one of the two pictures *LEFM* and *SFER* be closed, then the other also has to be closed, i.e. if the straight lines *LR* and *MS* should come together (meet) in one direction, then they must come together also in the other direction. However, since it is impossible that two straight lines meet at more than one point, it follows that the two pictures *LEFM* and *SFER* are open, i.e. the lines *LR* and *MS* do not meet in either direction, or *LR* // *MS*.[380]

By allowing movements of open figures, Asperén and Olson could change the outline of the theorems more freely. Take for instance the proof of Olson's Prop.17 above. Both Asperén and Olson merely had to use one theorem in their proofs, the theorem on supplement angles. If we backtrack to Lindman's proof of Prop.I.27, he needed the following theorems in Book I:

16, 15, 13, 11, 10, 9, 8, 5, 3, 2, and 1

Thus, the allowance of more types of movements and more types of objects being moved made it easier to accomplish a new outline of the theorems. Moreover, Prop.I.16 did not have to be included, a theorem that in some sense becomes superfluous when the students learn about Prop.I.32 and the fact that the exterior angle equals the sum of the interior and opposite angles.

> Prop.I.16. In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.[381]

> Prop.I.32. If a side of a triangle is produced, then the exterior angle equals the two interior and opposite angles. And the sum of the three interior angles of the triangle equals two right angles.

The exclusion of Prop.I.16 is another example of how Olson put pedagogical requirements before scientific. From a scientific point of view, Prop.I.16 is important since it applies for non-Euclidean geometries; which is not the case with Prop.I.32, since the proof requires the parallel postulate. However, non-Euclidean geometries were never a part of the courses and if we once consider the final exams students managed with just knowing Prop.I.32. Hence, as soon as Prop.I.32 had been established, students did not need Prop.I.16 to solve problems.

A difference between Olson and Asperén was that Olson included an explicit axiom on movements.

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[380] Olson (1940), pp. 25-26
[381] Vinell (1898), p. 31

> Hereby we assume as self-evident that lines and figures that we imagine to change position in space thereby do not undergo any kind of change with respect to shape and size (the axiom of movements, ax. 9).[382]

Olson did not, however, insert this axiom among the other axioms at the very beginning of the textbook. It was inserted in connection with the congruence theorems SSS and SAS.

Asperén and Olson and the reliance on illustrations and spatial intuition

In comparison to Asperén's textbook, Olson's treatment of school geometry is a bit more formalized. He formulated consequences of theorems that Asperén had tacitly assumed. He inserted an axiom on movements, and he described the properties of a symmetry line, even though he did not explain these properties via a formal definition or an axiom. Olson's axiom of movements was also an innovation in relation to the textbooks of Lindman and Vinell. Thus, Petrini's appeal to the textbooks authors to enhance the level of rigor, expressed in the early 1918, seems to have influenced Olson as well; even though they did disagree on several points. (See Part C of the thesis.) If we consider a passage in Olson's foreword, he directly pointed to Asperén and not Petrini as a source of inspiration.

> The book is to be considered a more consistent attempt to apply that which the author has found advantageous in these textbooks. To be mentioned in particular are the textbooks of K. Asperén, O. Josephson, and A. Meyer. [383]

However, despite the axiom on movements and the explicit formulation of the properties of symmetry, Olson relied on illustrations when he described the results of the movements. Without the illustrations, the reader cannot follow the proofs; take for instance Prop.11 (SSS congruence theorem) which is displayed above. Hence, on this point, Olson's textbook did not deviate from the textbooks of Vinell and Lindman. As a matter of fact, Asperén's proofs were more explicit in this respect; from his formulations it is clear where the object is placed. As I see it, this difference is linked to the fact that Asperén simply placed objects on top of other objects. Both Lindman and Olson also placed objects at the side another object. Moreover, just like Asperén, Olson did not justify all his inferences in the proofs. Also in this respect, he relied on the illustrations and spatial intuition.

In contrast, Sjöstedt provided a textbook that gave explicit descriptions regarding how objects were placed. He did insert illustrations, but when he

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[382] Olson (1940), p. 15. The italics are Olson's. This was Olson's axiom of movements.
[383] Olson (1940), pp. v-vi

moved objects, he also described, in words, how the objects were placed in relation to each other.

Sjöstedt's textbook

As mentioned above, Sjöstedt considered the Euclidean course to be a role model when authoring a textbook in geometry. However, the adherence to Euclid had to be weighed against pedagogical requirements.

> My position is that we should keep the essentials of Euclid's course and present in such a manner that no justified pedagogical requirements are set aside. … The course [of Sjöstedt's own textbook] is altogether that of Euclid; the textbook could therefore – if one so wishes – be seen as an abridged and revised edition of Euclid.[384]

In practice, this meant that he preserved the ordering of the propositions in the *Elements* to some extent, but by far not as closely as Lindman and Vinell did. Some propositions were left out, some new ones were added and the outline was changed. Sjöstedt also arranged the propositions thematically. In Appendix E, the propositions in the first chapter of Sjöstedt's textbook are listed.

Even though the changes were fewer and less radical in comparison to Asperén's and Olson's use of propositions on foldings and symmetry, Sjöstedt had to come up with new proofs, since the order of the theorems was changed. His approach to this problem was to use the congruence theorems more often. This was an essential concept to Sjöstedt; it was the core of a Euclidean course in geometry.

> One should be able to say that the most characteristic feature of this course is the central position of the theory of congruency. The first and second congruence theorems are inserted among the first propositions and are thereafter consequently applied.[385]

Sjöstedt also introduced new types of congruence theorems that required movements of other types of figures than only triangles. He also allowed movements where figures were turned over.

Another deviation from the *Elements* concerns axioms; Sjöstedt considered the changes of geometry at a scientific level so fundamental that even the axiomatic system in an elementary textbook had to be modified.[386]

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[384] Sjöstedt (1936), pp. 5-7
[385] Sjöstedt (1936), pp. 6-7
[386] Sjöstedt (1936), pp. 5-6

Sjöstedt's introduction of concepts and axioms

In comparison to the other Swedish textbooks of this time, Sjöstedt's introduction of axioms and basic concepts is most similar to Hilbert's style in the *Grundlagen der Geometrie*. Sjöstedt underscored that the point, the straight line, and the plane are the three most important concepts in geometry. These concepts were the simplest and therefore also impossible to define. Consequently, he did not present any kind of explicit definition of these concepts. Nor did he give any concrete examples, for instance a ruler or a table surface, which we find in Asperén and Olson. Such kinds of explanations were absent throughout the textbook.

The properties of points and straight lines were given in three axioms.

> Axiom 1. Through two points, only one straight line can run.

> Axiom 2. Through a point outside a straight line, only one straight line can run parallel to the latter.

> Axiom 3. If a distance in a figure changes size from one value to another, then it runs through every value between these.[387]

After these axioms, Sjöstedt added seven axioms that treated quantities in general and the relations *equality*, *greater than*, *smaller than* and the operations *increase* and *decrease*. (See Appendix C) The eleventh and last axiom only concerned geometrical magnitudes and congruency.

> Axiom 11. Quantities that can cover each other are equal in size.

However, Sjöstedt did not include any axiom on movements, although he applied such on several occasions. We may compare this to Hilbert's list of axioms, where such axioms are included. In the debate in *Elementa* in the late 1930's, Nyhlén considered this to be a problem. Sjöstedt's responded that since such an axiom was not needed for the logical deductions, it should not be included in the textbook.[388]

Nor did Sjöstedt include any axioms that corresponded to the first three Euclidean postulates on the generation of straight lines and circles. Through out the textbook, he tacitly assumed the possibilities of doing such constructions. He also assumed the possibility of moving straight lines and of cutting off a shorter straight line from a greater. The possibility of these constructions is established in Prop.I.2 & 3 in the *Elements*.

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[387] Sjöstedt (1936), pp. 12-13
[388] See Part C of this dissertation for the arguments of Nyhlén and Sjöstedt on this issue.

# Congruence theorems, movements and spatial intuition in Sjöstedt's textbook

As I have pointed out above, Sjöstedt considered the congruence theory to be the core of the Euclidean course. The most important propositions in this respect were the congruence theorems SAS and SSS; apart from the axioms, they were used most often. Sjöstedt also pointed out that one advantage of applying the congruence theorems as much as possible was that foldings were avoided.[389] However, Sjöstedt did use movements where figures were turned around or converted. The proposition on base angles in an isosceles triangle was proved in this manner (Prop.I.5 in the *Elements*), just as Asperén did.

The SSS congruence theorem was also proved by means of this type of movement.

> If each three sides in a triangle are equal to the sides in another triangle, then the angles are congruent.
>
> Given: In *ΔABC* and *ΔA´B´C´*, *AB* = *A´B´*, *AC* = *A´C´*, *BC* = *B´C´*.
>
> Claim: *ΔABC* ≅ *ΔA´B´C´*
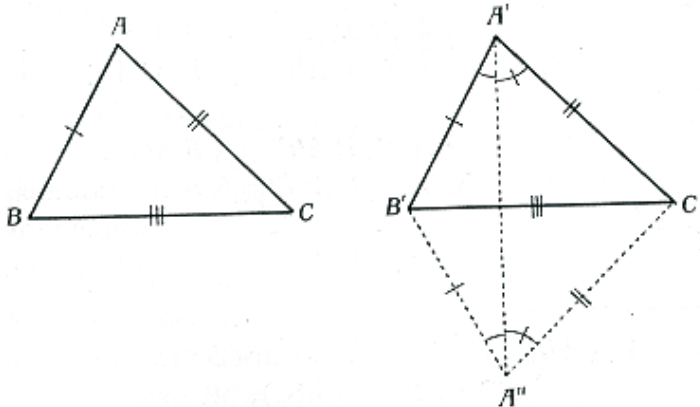
> Proof: Move *ΔABC* in such a manner that point *B* falls on point *B´* and the side *BC* falls along the side *B´C´* and point *A* falls in a point *A´´* on the opposite side of *B´C´* against point *A´*.
>
> The point *C* falls on the point *C´*, since *BC* = *B´C´* (given).
>
> Draw *A´A´´*.

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[389] Sjöstedt (1936), p. 7

Now $A''B' = AB$ (the same line in different positions)

But $AB = A'B'$ (given)

$\therefore \quad A''B' = A'B'$ (ax. 4)

$\wedge B'A'A'' = \wedge B'A''A'$ (prop. 5)   [prop.5: the base angles in an isosceles triangle are equal]

Moreover, $A''C = AC$ (the same line in different positions)

But $AC = A'C'$ (given)

$\therefore \quad A''C' = A'C'$ (ax. 4)

$\wedge A''A'C' = \wedge A'A''C'$ (prop. 5)

Now, we have proven that

$\wedge B'A'A'' = \wedge B'A''A'$ and that

$\wedge A''A'C' = \wedge A'A''C'$

$\therefore \quad \wedge B'A'A'' + \wedge A''A'C' = \wedge B'A''A' + \wedge A'A''C'$ (ax. 7)

or $\wedge B'A'C' = \wedge B'A''C'$.

But $\wedge B'A''C' = \wedge BAC$ (the same angle in different positions)

$\therefore \quad \wedge BAC = \wedge B'A'C'$ (ax. 4)

Hence, in $\varDelta ABC$ and $\varDelta A'B'C'$ we have

$AB = A'B'$ (given)

$AC = A'C'$ (given)

$\wedge BAC = \wedge B'A'C'$ (recently proven)

$\therefore \quad \varDelta ABC \cong \varDelta A'B'C'$ (1:st congruence theorem)   [i.e. the SAS theorem]

Q.E.D. [390]

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[390] Sjöstedt (1936), pp. 23-24

Note that Sjöstedt clearly describe how A'' is placed in relation to A and $B'C'$. We can compare this to Olson's proof, where Olson relied on the illustrations alone.

Sjöstedt proved the SAS congruence theorem in the same way as in Euclid's *Elements*, Heath's version.[391]

In line with the idea to use the congruence theorems as much as possible, Sjöstedt applied them even more often than Euclid did. One example is the construction problems that correspond to Prop.I.9-12 in the *Elements*, Heath's edition. At this point we can make a comparison with how both Asperén and Olson applied their theorems on foldings in the proofs of these problems.

However, Sjöstedt left out some of the theorems in the *Elements*. In order to cope with this, he introduced a new congruence theorem on open figures. Asperén and Olson applied movements of a similar type of figure, but they did not have a particular theorem for it, see Olson's Prop.17 above.

> Prop.14. If a distance in a figure equals another distance in another figure, and at the ends of the distances, mutually equal angles are allocated, and they [the angles] are placed in the same manner, then the two figures are congruent.
>
> Given: In the figures $CABD$ and $C'A'B'D'$, $AB = A'B'$, $\wedge A = \wedge A'$, $\wedge B = \wedge B'$.
>
> Claim: $CABD \cong C'A'B'D'$
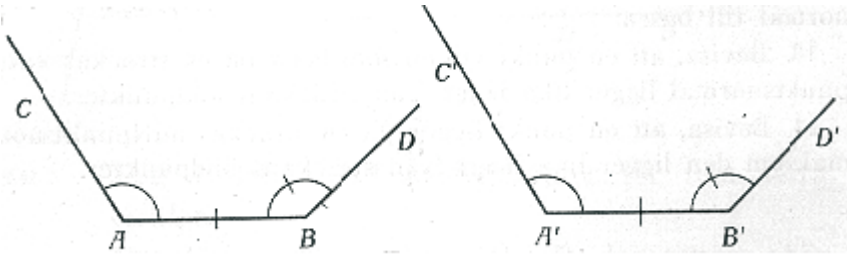
> Proof: Place $CABD$ on $C'A'B'D'$ so that A falls on point A' and the distance AB falls along the distance $A'B'$.
>
> Then point B falls on point B' , since $AB = A'B'$ (given).
>
> Furthermore, AC falls along $A'C'$, since $\wedge A = \wedge A'$ (given).
>
> Finally, BD falls along $B'D'$, since $\wedge B = \wedge B'$ (given).

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[391] Sjöstedt (1936), pp. 19-20, Heath (1956), vol. 1, pp. 247-248

Hence, the figure CABD covers the figure $C'A'B'D'$.

∴ the corresponding elements in the figures are equal. (ax. 11)[392]

∴ $CABD \cong C'A'B'D'$

Q.E.D. [393]

According to a footnote, lines CA, BD, $C'A'$ and $B'D'$ are indefinite or not restricted.

> The legs AC, $A'C'$, BD and $B'D'$ are not restricted, but they can be extended without restrictions.[394]

In comparison to Euclid's *Elements* (Heath's edition), but also the other textbooks investigated in this thesis, Sjöstedt's Prop.14 stands out as he applied indefinite, or infinite, lines in connection with congruence and in connection with expressions about elements being equal. In Prop.14, Sjöstedt actually established that infinite lines are equal.

In relation to Euclid's *Elements*, the use of infinite straight lines, or any infinite quantity for that matter, is highly unconventional. If we consider the fifth book in Euclid's *Elements* and the first definition, a part of a quantity is defined as a quantity that measures the greater quantity. Since a finite quantity cannot measure an infinite quantity, it is not meaningful to consider a finite quantity a part of an infinite quantity.[395] Consequently, it is not meaningful to consider finite lines as parts of infinite lines.

Was it then meaningful for Sjöstedt to say that a finite line is smaller than an infinite line, or to say that a finite line is a part of infinite line? On the basis of Sjöstedt's textbook, his standpoint in this matter is not clear. In connection with his axioms, mentioned above, he did not state explicitly whether or not they encompassed both finite and infinite quantities.

Another aspect of Sjöstedt's Prop.14 is that he uses it in a way that is hard to follow sometimes. By this new congruence theorem, Sjöstedt quickly proved the ASA-congruence theorem. Without further comments and illustrations, he considered it a special case of Prop.14.[396]
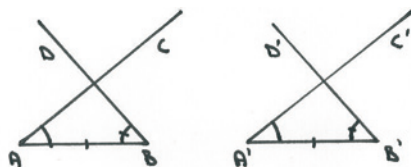
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[392] Sjöstedt's axioms are listed above.
[393] Sjöstedt (1936), pp. 34-35
[394] Sjöstedt (1936), p. 35: "Vinkelbenen AC, $A'C'$, BD och $B'D'$ äro icke begränsade utan kunna obegränsat utdragas."
[395] Heath (1956), vol. II, p. 113
[396] Sjöstedt (1936), p. 42

Yet, it is not obvious on what grounds Sjöstedt justifies that the pair of sides are mutually equal. If we follow the proof of Prop.14, it does not establish the equality of finite straight lines.

By means of Prop.14, Sjöstedt also could locate the propositions on parallel lines right after the chapter on the construction problems.

Prop.15. If two right lines are intersected by a third and a pair of alternate angles is equal, then the lines are parallel.

Given: *AB* and *CD* are intersected by *EF* in *E* and *F*. $\wedge a = \wedge c$.

Claim: *AB // CD*.
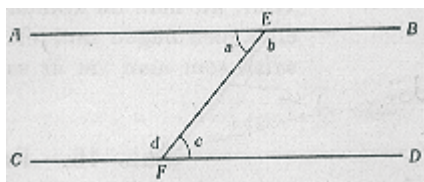
In the proof, Sjöstedt showed that $\wedge d = \wedge b$. Then, according to Prop.14, $AEFC \cong DEFB$. The last part of the proof goes:

If then *AB* and *CD* should intersect, e.g. in the direction of *A* and *C*, then they should intersect in the direction of *B* and *D* as well. The straight lines *AB* and *CD* should the have two points of intersection. This should be against axiom 1.

∴ *AB* and *CD* cannot intersect.

∴ *AB // CD*.   Q.E.D. [397]

In comparison, to the textbooks by Asperén and Olson, the textbook by Sjöstedt was closer to previous textbooks that followed Euclid's *Elements* quite closely, as for instance Lindman's and Vinell's. Especially when he put the traditional congruence theorems in the center and not some theorems that included foldings or symmetries. On the other hand, Sjöstedt deviated from the textbooks by Lindman and Vinell on several points. He included new axioms, and he did not insert any definitions of points, lines, planes etc; he

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[397] Sjöstedt (1936), pp. 36-37

was more generous in allowing different types of translations of figures; and the order of the theorems was quite different. Moreover, Sjöstedt allowed infinite lines in connection with congruence, which we do not see in the other textbooks.

Nyhlén's textbook – an experimental approach to axiomatic geometry

In the debate in *Elementa*, Nyhlén suggested that if it was not possible to uphold a scientific level of rigor in elementary geometry instruction, then the courses should be given more practical and experimental features. This idea was in some sense realized as Nyhlén published his textbook in elementary geometry in 1947. In this textbook, several of the propositions, but also the axioms, were given inductive and experimental proofs or explanations. Another innovation was the introduction of measures and real numbers. In the preface, Nyhlén underscored that this was done in order to simplify the chapters on parallelograms and circles. Moreover, the introduction of measures and real numbers was justified by the connection with needs in practical life.[398]

In the following subchapters, Nyhléns basic propositions and introduction of real numbers are described in more detail.

An experimental approach to axioms and proofs

Just like Sjöstedt, Nyhlén applied the style of Hilbert and excluded specific definitions of these concepts. The meaning of these concepts was given by a set of axioms. Nyhlén's axioms are listed in Appendix F.

A distinct feature of Nyhlén's treatment of axioms was that they were given an experimental introduction; the meaning of the axioms was linked to the use of two kinds of drawing tools: transporters of straight lines and angles.

> If *AB* is a distance, then we mark two points *C* and *D*, which coincides with *A* and *B*, on the edge of a transporter of distances (e.g. the edge of a folded piece of paper). If the points *C* and *D*, by an appropriate transportation of the transporter, coincide with the endpoints *A´* and *B´* of another distance, then the distances *AB* and *A´B´* is an example of two equal distances. In connection to constructions, the distance transporter is used to allocate a distance equal to a given distance. …

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[398] Nyhlén (1947), p. v

Basic proposition 2. If *AB* is a distance and *A´* is a point on a straight line, then you can determine exactly one point *B´*, on each side of *A´*, so that *A´B´* equals *AB*. …

If *AOB* is an angle, the legs of the angle transporter are thus placed, that they coincide with *OA* and *OB*. If the legs of the transporter, by an appropriate transportation, coincide with the legs *O´A´* and *O´B´* of another angle, then the angles *AOB* and *A´O´B´* are examples of equal angles. In connection to constructions, the angle transporter is used to allocate an angle equal to a given angle. ...

Basic proposition 5. If *AOB* is an angle and *O´A´* is a ray, then you can determine exactly one ray *O´B´* on each side of the line *O´A´*, so the $\wedge A´O´B´$ equals $\wedge AOB$.[399]

These two drawing tools were used to introduce and explain the eighth axiom, which is the SAS-congruency theorem.

Basic proposition 8. Construct an *Δ ABC*. Allocate $\wedge A´ = \wedge A$ and on the legs the points *B´* and *C´* so that *A´B´ = AB* and *A´C´ = AC*. Draw *B´C´*. Compare *BC* and *B'C'*, $\wedge B$ and $\wedge B´$ and $\wedge C$ and $\wedge C´$ by means of transporters. The result gives us reason to insert the following proposition:

If two sides and the adjoining angle in a triangle equal two sides and the adjoining angle in another triangle, then the triangles are congruent (1st case of congruence). [400]

An advantage of these axioms, Nyhlén argued, was that foldings and movements of triangles became obsolete. Recall that Nyhlén had criticized both Olson and Sjöstedt on this point. Moreover, if we consider Nyhlén's choice of axioms, axioms 2, 5, and 8 remind us of Hilbert's axioms that treat movements of straight lines and angles.[401]

By means of these basic propositions, he also proved, or explained, the proposition on isosceles triangles and their base angles.

Proposition 7. Construct an $\wedge C$ and allocate *CA = CB* on the legs. Draw *AB*. Compare $\wedge A$ and $\wedge B$ by means of a transporter. The result gives us reason to insert the following proposition.

If two sides in a triangle are equal, then the opposite angles are equal. (Proposition on base angles). [402]

[399] Nyhlén (1947), pp. 3-5
[400] Nyhlén (1947), pp. 8-9
[401] See for instance Hilbert (1913), pp. 2-23
[402] Nyhlén (1947), p. 13

Also the SSS and ASA congruence theorems were proved in a similar manner.[403]

However, in connection with the subsequent propositions on parallel lines, triangles and parallelograms, the axiomatic method was introduced. According to Nyhlén, his introduction of the axiomatic method was prepared in such a way that the students would not take notice of it.[404]

The use of real numbers

To Nyhlén, the introduction of real numbers and measures was a means to exclude several of the propositions in Euclid's *Elements*, but also a way to meet needs in practical life.

> Considering the needs of the practical life you give, as early as possible, the account an arithmetical form (se for instance proposition 29), which also bring very considerable simplifications. This requires a foundation where each distance and angle can be assigned a number. Thus, the first task that is brought to you in connection to the arithmetization of geometry is to insert measures of distances and angles on the basis of the basic propositions. When measures of distances and angles are inserted, the theory of surfaces and circle arcs can be arithmetized immediately. Purely geometrical propositions on surfaces of parallelograms and the Pythagorean Theorem as well as propositions on circle arcs are therefore completely superfluous. Moreover, a purely geometrical theory, based on special basic propositions on surfaces – e.g. Euclid I:35-48 – is now antiquated. At the level of Realskolan one can, of course, only give an extremely simplified account on the theory of measures. This theory is then extended and deepened at the Gymnasium.[405]

The elimination of Prop.I.35-48 in the *Elements* was done by the introduction of the area formula for rectangles. The area formula was justified and explained in same manner as in the textbooks for Folkskolan. By simple constructions, the area formulas for triangles, parallelograms, and parallel trapezoids were derived.
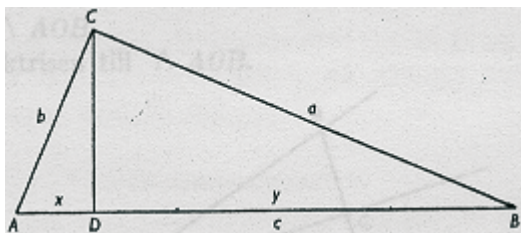
The Pythagorean Theorem, Prop.I.47, was included in the chapters on uniformity and proportions. In this way, the proof was made shorter and less complicated.

> Proposition 46. In a right triangle, the square on the hypotenuse equals the sum of the squares on the other two sides. (The Pythagorean Theorem).

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[403] Nyhlén (1947), pp. 13-14
[404] Nyhlén (1947), p. v
[405] Nyhlén (1947), p. v

Given: In $\triangle ABC$, $\angle C$ is right. CD is the height against the hypotenuse.

Claim: $a^2 + b^2 = c^2$.

Proof: $\triangle ADC \sim \triangle ACB$ (prop. 43), which among other things means that $x/b = b/c$.

$\triangle BDC \sim \triangle BCA$ (prop. 43), which among other things means that $y/a = a/c$.

The equality $x + y = c$ means that

$b^2/c = a^2/c = c$.

$\therefore a^2 + b^2 = c^2$.

To my knowledge, Nyhlén was the first to use this proof of the Pythagorean Theorem in a Swedish textbook in elementary geometry.

Concluding remarks – the textbooks in Realskolan and the significance of the professional debate

During the period 1905-1935, textbooks that followed Euclid's *Elements* very closely were fairly common in Realskolan.[406] Here, I am referring to the textbooks by Lindman and Vinell. They even named their textbooks *The first four books of Euclid*. In comparison to Heath's edition of Euclid's *Elements*, the theorems and the outline were the same, but some of the proofs were altered.

   Thus, quite a number of teachers seemed to side with Petrini in the choice of textbooks. The textbooks by Lindman and Vinell were used until the late 1930's, when they were excluded from the official list of approved textbooks, a list that appeared for the first time in 1935.

   As I have described in Part C of this dissertation, arguments against Euclid's *Elements* as a textbook was conveyed during the 1920's. Moreover, the benefits of alternative textbooks were also described. In this part, I have

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[406] In this case, I am referring to Heath's edition of Euclid's *Elements*.

shown that alternative textbooks were produced during the period 1905-1935. Here, I am referring to the textbooks by Asperén, Josephson and Olson. These textbooks share the following characteristics:

- They deviated from the Euclidean outline and the number of theorems needed for the proofs of some theorems was severely increased. This meant also that the theorems could be grouped more thematically. Some theorems were dropped completely.
- The definitions and all the axioms were not given via a list in the beginning of the textbooks. They were introduced, when needed.
- The authors did not always make explicit references to axioms and previous theorems.
- The Euclidean theorems on congruency were complemented by theorems on foldings and symmetry lines. In some proofs, these theorems also replaced the Euclidean congruence theorems.
- Movements of triangles, but also other types of surfaces, were applied more often then in Euclid *Elements*.

These characteristics correspond well to the arguments conveyed by Meyer, Hedström and Olson in the professional debate on geometry instruction.

However, the authors of the alternative textbooks did not abandon the axiomatic method. Nonetheless, as I just mentioned you can find proofs in the textbooks of Asperén and Olson where the authors did not give a formal justification to every inference. For instance, they did not always motivate explicitly that two straight lines that share two points coincide.

According to an investigation on mathematics instruction published in 1871, textbooks very similar to Euclid's *Elements* were the most common at the lower secondary level. The alternative textbook designed by Asperén was first printed in 1896. According to an investigation on textbooks in the Swedish schools published in 1931, the alternative textbooks were used by a clear majority of the teachers in Realskolan. Thus, we can observe a change in the choice of textbooks among mathematics teachers during the period 1905-1931. Since this change was not forced by official directives regarding textbooks, I think it fair to say that a majority of the teachers found the arguments like those of Meyer, Hedström and Olson to be the most relevant.

This change in the choice of textbooks provides interesting information about the professional debate about geometry instruction that was going on in the 1920's. Petrini's criticisms of contemporary textbooks and teaching were not directed against some recent trend. It was Meyer, Hedström, and Olson, not Petrini, who were the defenders of the prevailing situation.

After 1935, the textbooks by Olson and Sjöstedt were probably the most successful textbooks first published after 1925. This conclusion is based on the fact that their works were reprinted most frequently. In 1936, Sjöstedt published his first geometry textbook, which could be considered a design in accordance with Euclid's course, he claimed. In comparison with Lindman's and Vinell's editions of Euclid's *Elements*, Sjöstedt deviated from this

course on quite some points. He shortened the chain of theorems needed for some proofs; he dropped some theorems and inserted some new ones; he applied movements of triangles and other types of surfaces more often. Sjöstedt even used a congruence theorem that included infinite straight lines, which is unconventional in connection with classical geometry. However, he did not include theorems on symmetry or foldings; he inserted a list of axioms in the beginning of the textbook; and he preserved the Euclidean outline to a much greater extent than Asperén and Olson did.

Olson published his first textbook in geometry 1925.

Hence, up to 1965 the teachers mainly chose between the textbooks by Olson and Sjöstedt along with the older textbooks by Asperén and Josephson.

An important aspect is that all these textbooks were designed according to the axiomatic method. Consequently, as the students began to study geometry in grade 7, they were introduced to axiomatic geometry and an apparatus of definitions, axioms, theorems, and proofs. This provides us with an idea of what the arguments about training in reasoning in connection to geometry, but also mathematics in general, meant to the persons involved. The axiomatic method did constitute a central component this type of reasoning.

A third type of alternative textbook introduced by Nyhlén in the late 40's. This textbook had a more experimental approach, and Nyhlén applied real numbers in order to make the some proofs more attainable. However, this textbook was also predominantly designed according to the axiomatic method. Nyhléns textbook was not a success among the teachers.

Nonetheless, the arguments of Petrini and Nyhlén about raising the level of rigor and precision seem also to have been relevant to the authors of alternative textbooks. Olson, whose textbook was published in 1925, inserted axioms and comments which his role model Asperén did not insert in his textbook, e.g. an axiom on movements. Olson also explained the properties of foldings, which we do not find in Asperén's textbook. In Sjöstedt's textbook, first printed in 1936, we can see an ambition not to rely on illustrations as he moved geometrical objects; he explicitly described how the objects were placed. We can compare this to how Olson and Asperén quite often did not formulate how the objects were moved and placed.

On the other hand, the scientific ideal regarding rigor and precision expressed by Petrini and Nyhlén were not easy to live up to. In all the textbooks investigated, there are examples where the authors did not justify their inferences by means of previous theorems or axioms; instead they relied on illustrations and our intuitive understanding of the geometrical objects and their properties. Even Petrini can be criticized in this respect. As he re-edited Lindman's textbook in the early 1920's, he did improve crucial details in Lindman's proof of Prop.I.5. However, he did not explicate all the essential features of the theorem and the proof, which other textbook authors did, for instance Sjöstedt. About the same time Petrini argued that Euclid's *Elements*

was the ideal textbook, and he called upon others to make the *Elements* better with respect to exactness. Petrini argued also that a high level of precision would make the proofs easier to follow for the beginner.

This provides a hint about the trouble the textbook authors faced as they wanted to modify Euclid's *Elements* or develop alternative textbooks based on the axiomatic method. To achieve new proofs was not an easy task.

To end with, I think it fair to say that the supply of geometry textbooks for Realskolan during the period 1905-1962 was wide; the teachers could choose between distinct types of textbooks where the selection of theorems and the design of proofs were different. At this point we should not forget that the number of students in Realskolan increased and new schools were started throughout the period 1905-1962.[407] Thus, the textbook market was expanding. In that perspective, the professional debate about geometry instruction in Realskolan, which I describe in Part C of this thesis, was relevant to the teachers in the lower secondary schools. In this debate, the axiomatic method and textbook design were main issues.

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[407] See the background chapter on the Swedish school system.

Part F – Geometry exercises in final examinations in mathematics in Realskolan

A general description of the exam process

The final examinations for Realskolan constituted an ending at Läroverken for the students aiming for vocational training or working life. The final exams in Realskolan were therefore voluntary, and students who went on to the Gymnasium generally did not take the exam; they left Realskolan after year eight.

The exam process took place during ninth grade, the last grade in Realskolan, at the beginning of May each year. A first part comprised a written test in Mathematics, together with written tests in Swedish, German, and English. The tests in mathematics usually comprised eight, but on some occasions seven, exercises. In the test in Swedish, the students were supposed to write an essay on one out of a small number of topics. The test in German comprised a translation of a given text from Swedish to German or a reproduction of German text. The test in English comprised the same two alternatives. Between 1905 and 1927, the test in Swedish was to be finished in four hours; the tests in Mathematics, German, and English were to be finished in three.[408] With the new regulations of 1928 and onwards, the time for the test in Mathematics was extended to four hours.[409]

The second part of the exam process comprised at least four oral tests in some of the subjects of the last year of Realskolan. The oral tests were taken in groups of five to eight students and each test lasted for 30 to 45 minutes.[410]

Between 1905 and 1927, the written tests in Swedish and Mathematics were mandatory, while at least one of the tests in German and English were to be taken.[411] From 1928 and onwards, only the test in Swedish was mandatory; the students then had to take at least two of the other written tests.[412] Hence, the students could choose not to take the test in mathematics.

In order to proceed to the oral examination, the students had to pass the test in Swedish and at least one of the other tests. In 1928, this requirement

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[408] SFS 1905:6, p. 21
[409] SFS 1928:412, p. 1339; SFS 1933:109, p. 163
[410] SFS 1905:6, p. 21; SFS 1928:412, p. 1339; SFS 1933:109, p. 194
[411] SFS 1905:6, p. 20
[412] SFS 1928:412, p. 1338

was extended to at least two of the other tests. However, students who passed just one of the other written tests could still proceed if two thirds of the teachers approved of this.[413]

The correction processes of the exam in mathematics, but also the other subjects, comprised two parts. First, the tests were corrected by the teachers at the schools; thereafter, they were sent to one or more test examiners who corrected the tests once more. During the first decades of the 20th century, all tests from all schools were corrected a second time, but as the number of students increased, not all schools had to send in their tests. In the early 1950's, one of the test examiners made a comment that one third of the tests had been corrected once more.[414]

These examiners brought together statistics from the teachers' corrections and their own corrections; these were presented in a report together with comments on the results and the exercises. In the final summary of these reports, we also find comments about how some exercises ought to be corrected and how the teachers had carried out the corrections.

Until 1951, these reports contain extensive and detailed statistics, and the records are well organized. After 1951, the reports are much briefer and the records are not in good order, with parts missing as well.[415]

The final examinations in mathematics – a brief description and some statistics

From the beginning in 1905, Realskolan comprised only one program. In 1934, three practical programs were added: trade, technology, and household matters. The old program was then denoted the general education program. Yet, all students followed the same course program in mathematics, and they took the same tests in mathematics during the exam process – there were no special courses for some category of students. In 1951, different final exams in mathematics were given: one for the longer course and one for the shorter. The shorter course did not contain geometry exercises that required any sort of proof.

As I have mentioned, the students that entered the Gymnasium, i.e. the upper secondary level, usually left Realskolan in year eight, and they did not take the final exams in Realskolan. Hence, the test results that I display in this chapter do not reflect the capacity of all the students in Realskolan. Nonetheless, since all students, up to 1951, followed the same course program, the final exams in mathematics do reflect what the students in general were expected to know about mathematics when they left Realskolan. The

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[413] SFS 1928:412, p. 1339, SFS 1933:109, p. 163
[414] Riksarkivet A, vol. 75
[415] Riksarkivet A, Riksarkivet B

tests also provide good information about what kind of exercises the teachers included in their teaching.

The tests usually comprised eight exercises. The exercises could have the following topics.[416]

- Arithmetical and algebraic reductions
- Equations
- Equation systems
- Currency reductions
- Problems on profit and loss
- Discounts
- Percentage and interest rate
- Exchange bills
- Shares and bonds
- Company exercises
- Mixture exercises
- Working problems
- Uniform motion
- Number problems
- Plane geometry
- Stereometry

Before 1950, the exam test in mathematics contained one to three geometry exercises.

From 1950, the number of geometry exercises increased on the final exams in mathematics. The tests included, up to four exercises related to geometry. According to a circular from the central school authorities to the schools, the level of difficulty of the most difficult exercise should remain at the same level as during the previous decade, but the less difficult exercises was supposed to become easier. The teachers were also recommended to devote more lessons to geometry. Moreover, in the same circular the central school authorities declared that the general level of difficulty of the final exams in mathematics was to be lowered, not only the geometry exercises. This was to be accomplished by inserting more easy exercises.[417]

Each exercise on the final exam could render 1 or 0 points, i.e. satisfactory or not satisfactory. In order to pass the test in mathematics, the students had to achieve 3 points or more.[418] In the graphs below, we can observe the students' results on the tests in mathematics.

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[416] This categorization is taken from Stenbäck & Sundbäck (1962), a collection of exam exercises. Such collections were regularly published.
[417] Kungl. Skolöverstyrelsen (1950), s. 134
[418] Grimlund & Wallin (1939), p. 81

Figure 2.
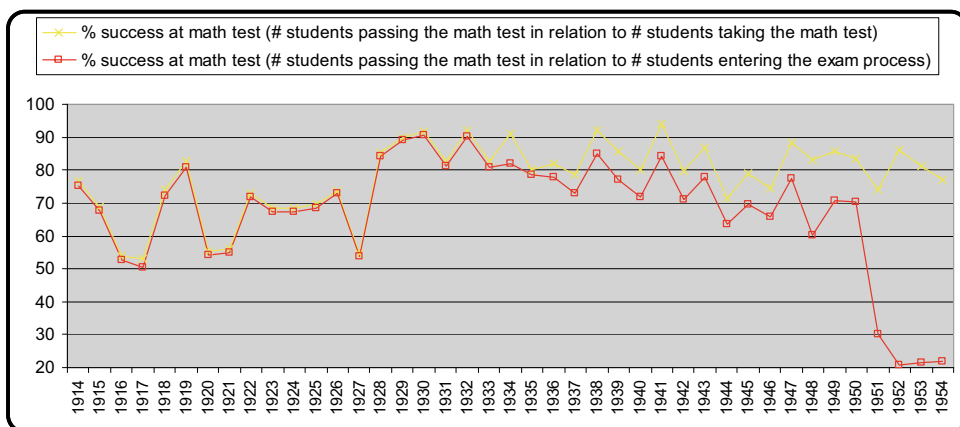
Here we can see that the extension of the time for writing the test by one hour from three to four hours, in 1928, coincides with an improvement of the results. Moreover, according to the graphs, the period 1933-1935 seems to have been a decisive moment for the students' achievements on the tests in mathematics. However, the trend is somewhat ambiguous. If the number of students succeeding is compared to the number of students that took the mathematics test, the results remain on a level around 80 percent or above up to 1954, even though there are some clear exceptions and we can observe a slightly negative trend. But, if the number of successful students is compared to the number of students that entered the exam process, the negative trend is clearer. The beginning of this trend coincides with the new curriculum of 1933, when the number of lessons in mathematics in Realskolan was decreased by approximately 20 percent.

Between 1935 and 1939, the share of students who took the test in mathematics during the exam process, decreased from about 98 percent to 90 percent. One could expect that this trend might have begun in 1928, when the test in mathematics became voluntary, but it did not. Furthermore, between 1928 and 1935, the number of students in Realskolan increased by more than 50 percent.

Figure 3.
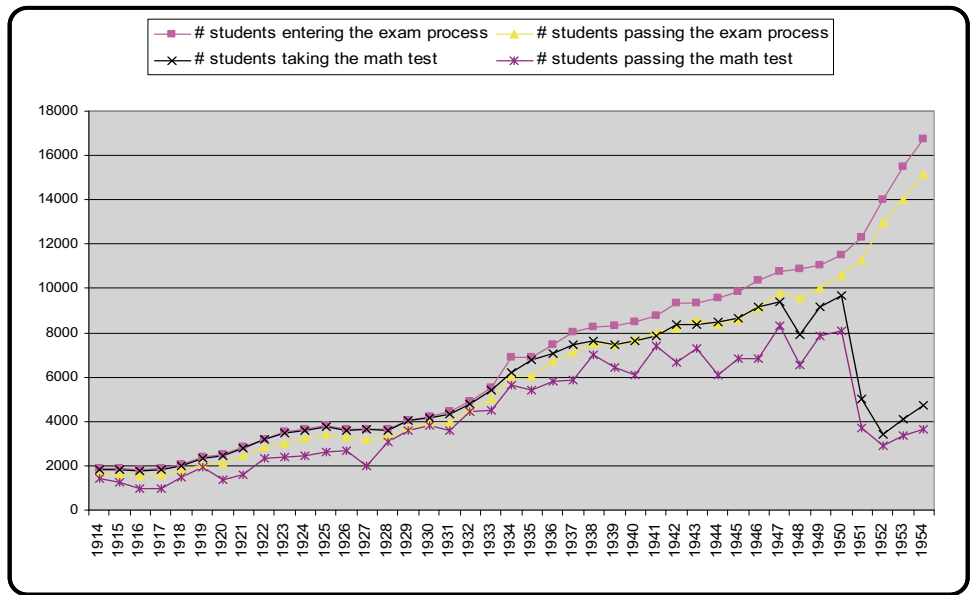
The dips in the lower graphs in 1948 might be caused by an incorrect transcript.

Figure 4.
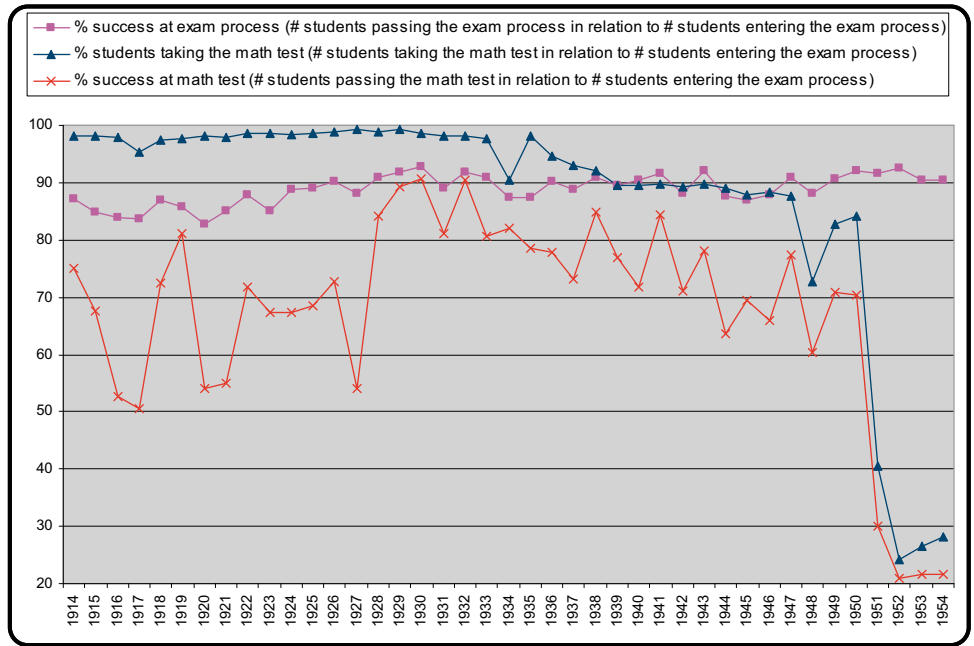
The dip in graph with triangles, 1934, is probably caused by an incorrect transcript.

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Thus, if we consider the graphs, it appears as if mathematics instruction between 1928 and 1934 functioned quite well, despite an increasing number of students and the fact that the final exam in mathematics was voluntary, almost 100 percent of the students took the test and more than 80 percent passed the tests.

The great decrease of the number of students that took the exam in mathematics, which begins in 1951, is most likely linked to the fact that students could choose to take an easier test in mathematics during the exam process.

The numbers in Figure 2-4 are based on Riksarkivet A, B and E, see the list of references.

The geometry exercises in the final examinations

Different types of geometry exercises and the requirements about reasoning

In the collection of exam exercises in mathematics (1917-1962) edited by Stenbäck & Sundbäck (1962), the geometry exercises were divided on four topics:

- Determination of angles
- The Pythagorean Theorem
- Uniformity (and proportions)
- Surface problems[419]

However, students were not informed about which category an exercise belonged to when taking the test.

Here follow some samples of geometry exercises from the final exams. The S indicates that the exercises were given in the spring. The number to the left of the colon indicates which year the exercise was given. The number to the right of the colon indicates which number the exercise had in the test. In some cases a number within parentheses is inserted, indicating that the same exercise was reused that year.

Determination of angles

> S 25:6 (S 52:7) When the opposite sides of an quadrangle inscribed in a circle are extended until they meet, one pair of the opposite sides generate an angle of 37° and the other pair an angle of 23°. What is the measure of the angles of the quadrangle?

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[419] In the exercises, the students are asked to compute a *surface*, not an *area* as we would say today.

S 30:6 In a triangle *ABC*, the sides *AB* and *BC* are equal. *P* is the centre of circle that is circumscribed around the triangle and *Q* is the center of a circle that is inscribed in the triangle. The angle *PAQ* is 6°. Compute the angles of the triangle.

S 39:5 (S 50:6) A ship takes a bearing at a beacon, i.e. one determines the angle between the direction of the ship and the line aiming at the beacon. The bearing is 40°. After 50 minutes of sailing where the course and the speed of 9 knots are kept, the new bearing at the same beacon is twice as great, i.e. 80°. How great was the distance, in nautical miles, to the beacon by the second time the bearing was taken? 1 knot = 1 nautical mile per hour.

The Pythagorean Theorem

S 24:8 A circle goes through the endpoints of one of the sides to a square and touches the opposite side in its midpoint. Compute the radius of the circle, when the side of the square is 3.2 cm.

Uniformity

S 25:8 The square *ABCD* is 2 m^2. *B* and the midpoint M at *AD* are joined. *BM* intersects the diagonal *AC* at *E*. How great are *AE* and *CE*?

S 39:8 (S 50:8) An isosceles parallel trapezoid is circumscribed about a circle with a radius of 6 cm. The longer parallel side is 4 times the shorter parallel side. The points in which the circle touches the equal and non-parallel lines are joined. Compute the length of this unification line.

S 45:8 In the triangle *ABC*, the height *AH* lies within the triangle. *BH* is 5 cm. *AH* is 1 cm shorter than *AB*. In addition, *AC* and *CH* are 36 cm together. In the circle that is circumscribed around the triangle *ABC*, a diameter *AD* is drawn; after that, *D* is joined with *C*. Compute the length of *CD*.

Surface problems

S 30:3 A field has the shape of a triangle *ABC*, where *AB* = 108 m, *AC* = 144 m and *BC* = 180 m. From a point *D* on *AB*, situated 48 m from *B*, we want to erect a fence *DE* parallel to *BC* and across the field. What is the length of the fence?

S 40:4 In a circle sector *ABC*, a circle is inscribed so that it touches the arc *BC* of the sector and the radii *AB* and *AC*. The point *O* is the centre of the inscribed circle, whose radius is 2 cm. *OA* is 4 cm. Compute the surface of the sector. π = 3.14.

S 40:5 A real estate area has the shape of a parallel trapezoid, where the two parallel sides are perpendicular to one of the other sides. On a map, with the scale 1:4000, the parallel sides are 1 cm and 2 cm, and the greater of the other

sides is 3 cm. How much is the real estate area worth if the price per square meter is 2.25 kr.

S 43:7 (S 52:6) In a parallel trapezoid each of the sides *AB*, *BC*, and *CD* is 6 cm and the fourth side *AD* is 12 cm. The diagonals *AC* and *BD* intersect in *O*. Compute the surface of the triangle *AOD*.

S 47:7 In an isosceles triangle, the base is 26 cm and the height against one of the equal sides is 24 cm. Compute the surface of the triangle.

S 54:7 In a right triangle *ABC*, the hypotenuse *BC* is 12 cm. From the midpoint *D* of the hypotenuse, a perpendicular *DE* is drawn. Point *E* lies on side *AC*. Compute the surface of the triangle *ABC*, when the distance *DE* is 2 cm.

Later on in this chapter, I will refer to some of these exercises.

Throughout the period, the solutions to all exercises were to result in the calculation of a number – the measure of an angle, the length of a line or the area of a surface. Moreover, the exercises contained measures. In this respect the exercises on the exam differed from the textbooks, which did not contain any measures at all. Thus, in the textbooks the axiomatic method and proofs were the main question; on the final exams, though, the students were supposed to apply their knowledge and skills in pure mathematics in order to calculate. However, we should not overestimate the importance of this difference. The exam exercises in mathematics were regularly published in booklets, which provided teachers with good opportunities to prepare students.

Apart from the categorization above, I discern two types of geometry exercises at the final exams that exhibit the following characteristics.

I　　　All the premises for the application of a numerical relation were given explicitly in the formulation of the exercise. A numerical relation was for instance the Pythagorean Theorem, the sum of angles in a triangle, and propositions on angles and circles. Hence, if the student knew the meaning of a couple of theorems and the numerical relations and was able to link these to the explicit premises, she could formulate a numerical or algebraic expression and then compute the angle, length, or area sought. The exercises often required one or more computations before the final computation could be carried out.

II　　　All the required premises for the application of the numerical relations were *not* given explicitly in the formulation of the exercise. Thus, the student had to make a more thorough investigation of the data, sort out which numerical relations may be useful, and find out what premises that are missing. Moreover, according to the comments in the exam report, a correct solution of these exercises should include some sort of justifi-

cation of the premises that the student was supposed to discover on their own. The test examiners regularly made comments about the lack of proofs and motivations. Like this one, from 1924: "... without having established any kind of proof ..."[420] The critical question is, of course, what type of justifications or proofs the students had to achieve. I return to this question below.

Note that the two categories above concern the geometry exercises on the final exams in Realskolan. There are similarities to my categorization of the exercises in the textbooks intended for Folkskolan, the type-A and type-B exercises, but we must remember that the exercises in Realskolan were much more difficult. In the type-A exercises, the students merely had to choose a formula out of a few and carry out one computation. In solutions to the type-I exercises the students had to choose one or more theorems out of a great number; moreover, the solutions required more complicated computations with more steps. Another difference is that some sort of proof was not required in Folkskolan.

Each exam test in mathematics during the period 1905-1962, except for one year, included one or two Type-II exercises. They were often given number seven or eight in the final exams. The Type-I exercises were common during the 1930's, but quite rare before that period and even rarer afterwards. Generally, 50 percent or more of the students could achieve a satisfactory solution to the Type-I exercises. However, there were exceptions. If we consider the solution frequencies of the Type-II exercises, they were generally below 50%; in half of the cases, the frequency was 30% or less. I return to this statistics later on.

The Type-I exercises includes exercises like S 30:3 and S 40:5 in the list above. They often had some allusion to practical applications. In this group, I have also included exercises, like S 25:6, which had no explicit connection to practical matters. In order to solve this exercise you have to apply theorems like I.32 and III.22 in Euclid's *Elements*[421] and all the premises are given explicitly in the exercise. In this example, the justifications are based on Olson's textbook, first published in 1925.

> S 25:6 When the opposite sides of a quadrangle, inscribed in a circle, are extended until they meet, one pair of the opposite sides generate an angle of 37° and the other pair an angle of 23°. What is the measure of the angles of the quadrangle?

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[420] Riksarkivet A, vol. 5
[421] Heath's edition
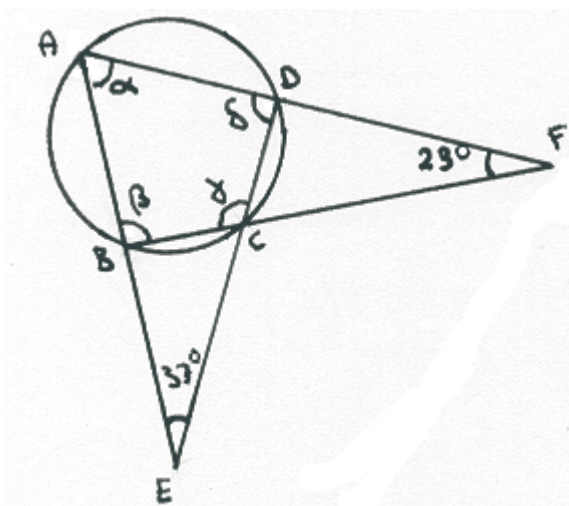
The angles at *A*, *B*, *C* and *D* are denoted α, β, γ and δ.

From Prop.82[422] we know that the opposite angles in a quadrangle inscribed in a circle are supplementary angles. From Prop.2 we know that supplementary angles equal two right angles. From this we get

$\alpha + \delta = 180°$ and $\beta + \delta = 180°$.

According to Prop.25[423], we also know that the sum of the angles in a triangle equals two right angles. From this we get

$\alpha + \delta + 37° = 180°$ and $\alpha + \beta + 23° = 180°$.

We can now determine α, β, γ and δ by solving the equations above.

Among the Type II exercises, we find exercises like S 24:8, S 25:8, S 30:6, S 39:5, S 39:8, S 40:4, S 43:7, S 45:8, S 47:7, and S 54:7. The most common relations that you have to recognize before you can formulate an algebraic expression are the following:

- Two angles are equal.
- One angle is a right angle.
- Two sides are equal.
- Two sides having a certain ratio.
- One angle is twice as great as another angle.

In connection with this, four types of theorems are useful.

[422] See Prop.III.22 in Euclid's *Elements*, Heath's edition.[Heath (1956), vol. 2, pp. 51-52]
[423] See Prop.I.32 in Euclid's *Elements*, Heath's edition.[Heath (1956), vol. 1, pp. 316-317]

- Theorems on straight lines and angles such as the theorems on vertical and alternate angles and the theorems on base angles and sides in isosceles triangles.
- Congruence theorems
- Theorems on uniform triangles and proportion.
- The theorems on angles, straight lines, and circles in Book III of Euclid's *Elements*[424]. Useful theorems were 18-22, 26-27 and 31-32.

The fact that the theorems on congruency were so crucial in connection to the final exams is interesting when we consider the design of the alternative textbooks. In these, new theorems on perpendiculars, symmetry, and foldings sometimes replaced the Euclidean congruence theorems. Hence, these new theorems were not just a technical innovation in the sense that they were to help the authors to construct textbooks where the proofs and the chains of theorems were shorter; they really took aim at theorems that the students had to master in order to solve exercises.

Here are two examples of how some premises must be established before you can formulate a numerical or algebraic expression. This time, the theorems are taken from Sjöstedt's textbook, first published in 1936.

S 25:8 The square *ABCD* is 2 m^2. *B* and the midpoint M on *AD* are joined. *BM* intersects the diagonal *AC* in *E*. How large are *AE* and *CE*?
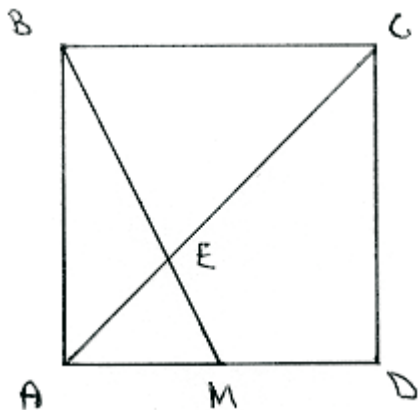
ABCD is a square. Its area is 2 m^2. *AM = MD*.

In order to determine the lengths of *AE* and *CE* you first have to determine their ratio.

We then have to establish that $\triangle AME \sim \triangle CBE$.

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[424] Heath's edition

∧*AEM* = ∧*CEB* since they are vertical angles (Prop.2)[425]. Since *ABCD* is a square, *AD* // *BC*. Hence, ∧*AMB* = ∧*CBM* and ∧*MAC* = ∧*BCA* (Prop.19)[426]. Since the corresponding angles in *ΔAME* and *ΔCBE* are equal, we have that *ΔAME* ~ *ΔCBE* (Prop.73)[427].

Due to the uniformity we have the proportion *MA:AE::BC:CE* (Def. uniformity). Since *BC = 2 AM*, we get the equality *CE/AE = 2/1*.

V 45:8 In the triangle *ABC*, the height *AH* lies within the triangle. *BH* is 5 cm. *AH* is 1 cm shorter than *AB*. In addition, *AC* and *CH* are 36 cm together. In the circle that is circumscribed around the triangle *ABC*, a diameter *AD* is drawn; after that, *D* is joined with *C*. Compute the length of *CD*.
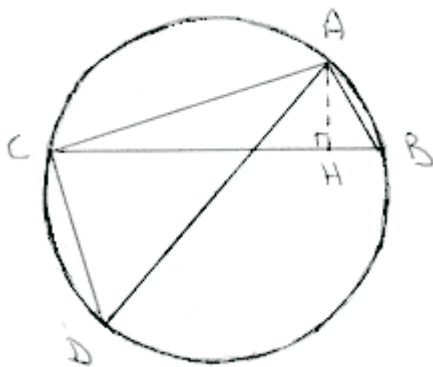
An important step in the solution is to establish that *ΔAHB* ~ *ΔACD*. Hence, we have to show that two pairs of corresponding angles are equal.

∧*ACD* = *R* since ∧*ACD* stand on the diameter *AD* and its vertex lies on the periphery (Prop.39)[428]. Thus, ∧*ACD* = ∧*AHB*, since *AH* is a perpendicular to *BC*.

∧*CDA* = ∧*CBA* since the angles stand on the same arc *CA* and their vertexes lies on the same periphery (Prop.40)[429]. Hence, ∧*CDA* = ∧*HBA*.

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[425] See Prop.I.15 in Euclid's *Elements*, Heath's edition.[Heath (1956), vol. 1, pp. 277-278]

[426] See Prop.I.29 in Euclid's *Elements*, Heath's edition.[Heath (1956), vol. 1, pp. 311-312]

[427] Prop.73: "If two angles of a triangle equal two angles of another triangle, then the triangles are uniform."[Söstedt (1936), pp. 96-97]

[428] Prop.39: "If a midpoint angle and an angle at the periphery stand on the same arc, then the midpoint angle is twice as large as the angle at the periphery."[Sjöstedt (1936), p. 61] In Sjöstedt's textbook this theorem encompassed midpoint angles equal to 2 right angles. In Euclid's *Elements*, Heath's edition, there is a separate theorem that establishes that if an angle at the periphery is standing on the diameter of the same circle, then the angle at the periphery equals 1 right angle.[Heath (1956), vol. 2, pp. 61-62]

[429] See Prop.III.21 in Euclid's *Elements*, Heath's edition.[Heath (1956), vol. 2, pp. 49-50]

Since two pair of angles is equal, $\Delta AHB \sim \Delta ACD$. (Prop.73)[430]

Obviously, these are my solutions and I do not claim that this was how the problems were solved. I merely wish to exemplify how some exercises required a more careful investigation of the formulation of the exercise. An important aspect is that students had to discover some premises on their own before they could formulate a numerical or algebraic expression, by which they could compute the answer. Thus, from a heuristic point of view, I think that the geometry exercises, especially the Type-II exercises, were quite demanding.

The final exams and the theorems in the textbooks

All the textbooks of the period 1905-1962 contained a set of theorems that were necessary for the solutions of the exercises on the test. The theorems were the following. The numbering is taken from Euclid's *Elements*, Heath's edition.

Book I: 4-6, 8-12, 14, 15, 26-30, 32, 47

Book III: 18-22, 26, 27, 31, 32

Moreover, all textbooks contained theorems on triangles, uniformity, and proportions, which also were required when solving the exercises on the final exams.

   Apart from these theorems, the textbooks that deviated from the *Elements* also contained new theorems. As I have described above, they were often inserted in order to bring about a new outline of the theorems. But some of the new theorems may have been useful when solving the geometry exercises on the final exams. At this point, we should also note that the Type-II exercises required that students were able to recognize certain premises before they could formulate an algebraic expression. For instance:
   - Two angles are equal.
   - One angle is a right angle.
   - Two sides are equal.
   - Two sides having a certain ratio.
   - One angle is twice as great as another angle.

In that context, the following theorem can have been quite useful. It establishes that one angle equals a right angle, the equality of two angles and the equality of two lines.

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[430] Prop.73: "If two angles of a triangle equal two angles of another triangle, then the triangles are uniform."[Söstedt (1936), pp. 96-97]

> Prop. 63 The tangents from a point to a circle are equal. The central line bisects the angles between the tangents, and it is the midpoint perpendicular to a chord between the end of the tangents.[431]

This particular theorem is taken from Sjöstedt's textbook, but we also find it in the textbooks by Asperén and Olson.[432] If we consider one of student papers displayed below, we can see that the student used this theorem.

A second new theorem, which may have been useful in the final exams, concerns circles and circumscribed quadrilaterals.

> Prop. 68 If a circle is circumscribed by a quadrilateral, the sum of two opposite sides equals the sum of the other two.[433]

This theorem is also taken from Sjöstedt, but Asperén and Olson inserted similar theorems as well.[434] On some occasions, the geometry exercises on the final exams contained such circles.

If we consider the student papers and the reports from the final exams, the students used a relation linked to the so-called 30°-60°-90° triangle. They then inferred that the side opposite the right angle is twice the side opposite the 30°-angle. This procedure was also accepted by the teachers. However, there is no such theorem in the textbooks. I suspect that this relation was established in connection with equilateral triangles. If we draw a bisector to one of the angles, the bisector will divide the opposite side in equal parts and we attain two 30°-60°-90° triangles. In one solution of a student that is displayed below, the student has applied this relation.

Policies regarding the correction of the geometry exercises

According to the reports of the examiners, the first official correction forms for the final exams in mathematics appeared in 1943.[435] Between 1944 and 1962, Sjöstedt was responsible for the official correction forms.[436] However, the comments made by the test examiners in the test reports have compensated for the lack of correction forms.

To my knowledge, there were no official and detailed directives that described what a proof should look like on the final exams. The official correction forms did not contain such directives. Most likely, the proofs in the

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[431] Sjöstedt (1936), pp. 84-85. Actually, the first proposition "the tangents from a point to a circle are equal", was proved in the proof of another theorem in Lindman's and Vinell's textbooks, but they did not give it a separate theorem.[Lindman (1894), p. 90; Vinell (1907), p. 108]

[432] Olson (1940), pp. 70-71; Asperén (1928), pp. 70-71

[433] Sjöstedt (1936), pp. 89-90. This theorem is proved by means of Prop. 63.

[434] Olson (1940), p. 81; Asperén (1928), pp. 76-77

[435] The first comment by the test examiners on some kind of official directive is made in 1943.

[436] Riksarkivet A, Riksarkivet B

textbooks constituted a model in this respect. At this point we have to re-member that some proofs in the textbooks were not perfect from a logical point of view. Some conclusions were drawn without references or with incorrect references. (See the previous part.) On the other hand, the textbook authors never abandoned the axiomatic method; their aim was to base the inferences on axioms and previous theorems.

However, there was room for proofs or justifications that did not meet the textbook standards. As pointed out above, you could receive 1 or 0 points on each exercise, i.e. pass or not pass, but there were two passing grades: *satisfactory* or *satisfactory with hesitation*.[437]

From 1949 onwards, the directive of the correction forms was that markings in the diagrams should be accepted as justifications, but then with the grade *satisfactory with hesitation*.[438] A recommendation in the correction form of 1956 had the following formulation:

> In order to receive "Satisfactory" there should be an acceptable justification for the generation of a 30°-60°-90°-triangle. If a written justification is not given or the formulation is defective, but from the diagrams and the calculations in general it is obvious that the student has understood the exercise, then the solution should be accepted with hesitation.[439]

What the policy on proofs or motivations looked like before 1949 is not as clearly stated. In the annual reports about the final exams, there are two comments where the examiners, in writing, accept markings as a motivation. About exercise S 25:8 above, the examiner made the following comment.

> The unsatisfactory thing about their solutions is, in the most cases, that no proof or a very incomplete proof is given for the claim that the line *BM*, from the corner of the square to the midpoint *M* of the side *AD*, divides the diagonal by the proportion 1:2. This proof is most conveniently achieved by the uniformity of the triangles *AEM* and *BCE*. I have considered solutions without any kind of proof for the uniformity of the triangles <u>not satisfactory</u>. But, I have in several cases been content with equal markings of equal angles as an indication of uniformity. In some cases, the examinees have achieved approximations of *AE* and *CE* by means of measurements. Such solutions I have judged as <u>not</u> satisfactory.[440]

[437] Riksarkivet A, Riksarkivet B. The grades *satisfactory* or *satisfactory with hesitation* were used by the test examiners throughout the period.

[438] Riksarkivet C, Riksarkivet D

[439] Riksarkivet D: "För "Nöjaktigt" bör det föreligga en fullgod motivering för att en 30°-60°-90°-triangel uppkommer. Om skriftlig motivering ej givits eller är bristfälligt formulerad men det av figuren och uträkningarna i övrigt synes framgå, att eleven förstått uppgiften, torde lösningen kunna godtagas med tvekan."

[440] Riksarkivet A, vol. 6: "Det otillfredsställande i deras lösningar är i de allra flesta fall, att intet bevis eller mycket ofullständiga sådana givits för påståendet att linjen BM från kvadrat hörnet till sidan AD:s mittpunkt M delar diagonalen i förhållandet 1:2. Beviset härför sker enklast medelst likformighet mellan trianglarna AEM och BCE.Jag har ansett, att en lösning utan bevis i någon form för trianglarnas likformighet <u>icke</u> är <u>nöjaktig</u>. Men, jag har i flera fall

There is similar remark about exercise S 37:8:

> Most solutions are based on uniformity, which the author has recognized but not often proved. This has been accepted without any further considerations.[441]

Moreover, in three other comments from the 1920's and 1930's we get the impression that vaguer types of justifications were allowed. The examiners complain about students who make assumptions without *any* motivation or without *further ado*.[442] My point is that they do not ask for a specific type of motivation, but *any* type of motivation. Another circumstance is that the examiner, in 1934, made a general comment on the level of formalism, which indicates that less formal proof should be accepted.

> The formal performance of the tests in mathematics is in general commendable. Yet, exceptions do occur in excessively great numbers. Since the written exam test in Realskolan explicitly is termed a "calculation test," not a test in mathematics, the examiners do not find it justified, more than very little, to consider the formal aspects of the test.[443]

What the examiner is referring to in the passage "explicitly is termed a 'calculation test'" is probably the fact that the tests were called "provräkning" in the school regulations, i.e. test in calculation.[444]

It might be that the two explicit recommendations on markings were exceptions to a policy where only textbook proofs were accepted. On the other hand, why should markings be accepted in these cases, but not in others? A more probable hypothesis is that the examiners found the correction of these two exercises a bit ambiguous and therefore wanted to specify a policy where markings in the diagram were regularly accepted. That hypothesis is also supported if we consider some of the actual student papers and their

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nöjt mig med att, att likformighet blott antytts genom markering av lika vinklar medelst lika beteckningar. I några fall har examinanderna genom mätning erhållit approximativt värde på AE och CE. Sådana lösningar har jag också bedömt såsom <u>icke</u> nöjaktig."

[441] Riksarkivet A, vol. 26: "De flesta lösningar stödja sig på likformighet, som förf. insett men ofta ej bevisat, något som utan vidare godtagits."

[442] Riksarkivet A, vol. 1 (1920) "De flestas fel är att de utan vidare göra det oriktiga påståendet, att kvadratens diagonal är normal mot hypotenusan och därpå bygger sin lösning."; vol. 5 (1924) "De flesta av dem, som misslyckats, ha utan att hava presterat något bevis, grundat sin lösning på det oriktiga påståendet, ... "; vol. 6 (1925) " De flesta, som fått felaktigt resultat, ha utan vidare gjort det oriktiga antagandet, ..."

[443] Riksarkivet A, vol. 18: "Provräkningarnas formella utförande är i regel berömvärt. Undantag förekomma dock i väl stor mängd. Då emellertid det skriftliga provet i real(skol)examen uttryckligen betecknas som "provräkning", icke som matematisk skrivning, ha eftergranskarna icke ansett sig berättigas vid valörsättningen att annat än i mycket ringa mån fästa avseende vid den formella sidan av provet."

[444] Grimlund & Wallin (1939), p. 80

corrections.[445] Here, we find corrections where the teachers accepted very simple justifications. They could consist of markings in the diagram or algebraic expressions that revealed that the student had recognized the meaning of the theorems. Here follows three examples of Type-II exercises from 1930, 1920 and 1910.

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[445] I have investigated the student papers at one of the secondary schools in Uppsala during the years 1907-1964. The school is Uppsala Högre Allmänna Läroverk, or Katedralskolan as it is named today. See "Landsarkivet, Uppsala", Uppsala" in the list of references.

S 30:6 In a triangle *ABC*, the sides *AB* and *BC* are equal. *P* is the centre of a circle that is circumscribed around the triangle and *Q* is the center of a circle that is inscribed in the triangle. The angle *PAQ* is 6°. Compute the angles of the triangle.[446]

This is the complete solution. In the centers of the two circles, there are two symbols *P* and *Q*. *Q* lies above *P* in the diagram. In this solution, the student overlooked the possibility of a second solution. In that case, the center of the smaller circle lies below the center of the larger circle. The teacher made a comment on this, but graded the solution "Satisfactory".

[446] Livius & Stenbäck (1941), p. 49

An important part in the solution is that the lines *AQ* and *QC* bisect the angles *BAC* and *BCA*. This is not motivated by the student; he not even marked the equalities of the angles *BAQ* and *CAQ* and the angles *BCQ* and *ACQ*. However, by putting the angles *BAC* and *BCA* equal to *x* and the angles *PAC* and *PCA* equal to *(x/2 – 6)*, he appears to have recognized these equalities. He also recognized the equality of base angles in isosceles triangles. From the theorem on the sum of angles in a triangle, he attains that the angle *APC* equals *180 – 2(x/2 – 6)* and the angle *ABC* equals *180 – 2x*. From the theorem on angles at the center and the circumference of a circle[447], the student formulates the expression 180-2(x/2-6)=360-4x, by which he solved the problem.

The fact that he used the equalities of the angles *BAQ* and *CAQ* and the angles *BCQ* and *ACQ* is interesting, since there are no theorems in Euclid's *Elements*, or the textbooks that followed the *Elements* closely, that establish this equality. However, in the textbooks that deviated from the *Elements* such a theorem was included. I have mentioned this theorem in a section above.

The following two solutions to S 10:7 and S 20:7 were also considered correct. Here, numerals were taken as an indication of that the students had understood the theorems.

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[447] In a circle the angle at the center is twice the angle at the circumference when the angles have the same circumference as their base.

S 20:7 An isoceles triangle is inscribed in a circle. The base of the triangle equals the radii of the circle (2 cm). Compute the angles and the surface of the triangle. More than one solution can be attained.[448]
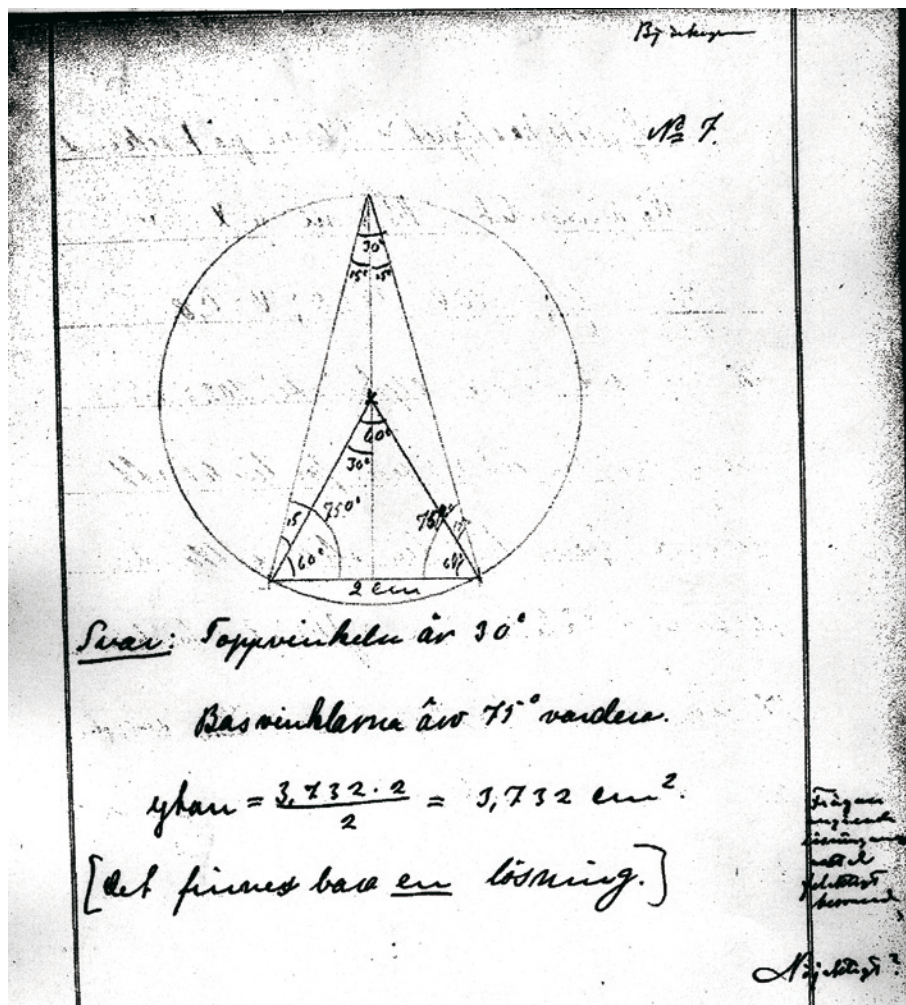
In this solution, which is the complete solution and not only the answer, the student only had to put out the measurements of various angles, which reveals that he had understood the theorems. He recognized that the smaller triangle is equilateral and its angles equal 60°. He recognized also that the top angle of the greater triangle is half the angle at centre of the circle, i.e. 30°. Since the greater triangle is an isosceles triangle, the base angles are equal and they must then equal 75°.

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[448] Livius & Stenbäck (1941), p. 29

S 10:7 In a triangle *ABC*, angle *A = 30°*, angle *B = 45°* and the height from *C* against *AB = 4 dm*. Compute the sides and the surface of the triangle.[449]

a)

Ant: den största sidan $(x + y)$ dm den minsta y dm.

$8\,dm$

$64 = 16 + x^2$

$x^2 = 48$

$x = \sqrt{48}$

$x = 6,928$

$y^2 = 4^2 + 4^2$

$y^2 = 32$

$y = 5,657.$

Svar: sidorna äro:

$10,95\,dm, 8\,dm, 5,66\,dm.$

b) Ant: ytan $= z\,dm^2$.

$\dfrac{10,95 \cdot 4}{2} = z$

$z = 21,86$

Svar: $21\,dm^2\ 86\,cm^2$ rätt

In this example also, the teacher accepted a solution where the student by measurements and algebraic expressions revealed that he had understood the meaning of the theorems. The student did not have to give a formal justification of the measurements of angles and lines. For instance, after having recognized that the smaller triangle containing the side *AC* is a 30°-60°-90° triangle, the student simply marked the line *AC* equal to 8 dm

These examples from three different decades provide a good idea about what markings could look like and what types of justifications that were accepted.

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[449] Livius & Stenbäck (1941), p. 9

Apart from the comments on markings, the exam reports also reveal solutions based on special cases, e.g. the students assumed that a triangle had a straight angle even though this was not explicitly established in the exercise. Sometimes, such solutions were approved by a majority of the teachers as well as the test examiners; sometimes, just by the teachers, sometimes by neither of them.[450] In the actual exam papers, we can also observe that the teachers accepted minor computational errors of the type *2 + 2 = 5* in the solutions to the geometry exercises.[451]

Considering the correction forms, the exam reports, and the corrections of student papers, the most important criteria for a correct solution of the geometry exercises on the exam was not any requirement regarding formal justifications or proofs. The important thing was that the students had understood the meaning of the theorems and that they could apply these theorems in connection with calculations. Another aspect is that when toiling with the geometry exercises on the final exams, it was more important to be a skilled problem solver than to be a master of formal proofs. If you could not discover the missing premises and if you could not come up with a suitable numerical or algebraic expression, you could not reach a satisfactory answer to the exercises. Moreover, all exercises asked the students to find a number.

In that respect, the teachers' reality and the arguments on the axiomatic method and the value of training in reasoning that we find in the articles on geometry instruction are quite remote from each other. I think that this circumstance adds an important aspect to our understanding of the relevance of the professional debate and the arguments about geometry instruction and training in reasoning.

Complaints about the achievements of students and teachers

In the reports on the final exams, complaints about the formal treatment of the geometry exercises occurred a few times during the 1920's and 1930's, mostly as very brief comments. From 1937 and onwards, these complaints appeared more frequently, and the formulations are longer and contain more severe criticisms. This is a remark on the solutions to exercise S 40:4.

> The treatment of exercise 4 displays how the students in Realskolan of today have difficulty with executing logical proofs, even if , as in the present case, they are of the simplest type. To this exercise, as to the exercises 5 and 8, the drawings accomplished are, on several occasions, most insufficient; indeed, in several cases the solution is not accompanied by any drawing at all.[452]

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[450] Riksarkivet A, vol. 1, 5, 6, 7, 10, 17 and 70.

[451] Landsarkivet, Uppsala, Uppsala

[452] Riksarkivet A, vol. 35: "Behandlingen av uppgift 4 ådagalägga svårigheten för realskolans elever att nu för tiden genomföra en logisk bevisföring, även om denna, som i ifrågavarande fall, är av enklaste art. I denna uppgift såväl som i uppgifterna 5 och 8 äro de utförda ritningarna många gånger synnerligen bristfälliga, ja, i många fall åtföljes lösningen ej alls av någon figur."

From about the same time, we can observe another type of complaint regarding the geometry exercises. These complaints did not concern the students' performances, but the teachers' ability to correct the geometry exercises. There were two types of criticisms: 1) some teachers tended to accept solutions based on incorrect justifications or no justifications at all; 2) some teachers appeared not to have understood the problem in the exercise at hand.

> The simple geometrical exercise 6, to which the justification of the equality of the sides of the triangle is essential and the most natural, has, in surprisingly many cases, been accepted either without or with insufficient or, indeed, even with incorrect justifications.[453]

In this case, the examiner was referring to exercise V 50:6. An almost identical exercise was used in 1939. In 1939, the test examiner did not make any comments like this at all. In a comment on exercise V 44:8, the examiner complained about some teachers' ability to understand the problem at hand.

> In connection with exercise 8, many solutions have been approved that include special treatments, right-angled triangles and special assumptions about the lengths of the sides. Thereby, the teacher in question has been shown not to have understood the meaning of this pretty geometrical problem.[454]

The complaints about the teachers did not occur every year; between 1945 and 1947, there were no complaints at all. However, from 1948 to 1952, they occurred each year. Between 1953 and 1962, there are no such comments, but that might be related how the reports were designed. They were much briefer after 1952.

In connection with these complaints about the teachers' abilities to understand and correct the geometry exercises, we can note that Sjöstedt and others, in articles, passed comments on the shortage of educated teachers in mathematics and the negative consequences of this shortage. Sjöstedt referred to an investigation that showed that more than 40 percent of the mathematics teachers in Realskolan did not have an adequate education in mathematics; their pedagogical education and experience was considered even more insufficient.[455]

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[453] Riksarkivet A, vol. 68: "Den enkla geometriska uppg. 6, i vilken helt naturligt motiveringen för triangelns likbenthet är det väsentliga, har i överraskande många fall godtagits såväl utan, som med bristfällig, ja, t.o.m. felaktig motivering."

[454] Riksarkivet A, vol. 47: "För uppgift 8 hava i många fall godtagits lösningar med specialbehandling, rätvinklig triangel eller specialantaganden om sidornas längder. Därigenom har vederbörande lärare visat sig ej förstå innebörden i detta vackra geometriska problem."

[455] Hilding (1958) p. 43; Sjöstedt (1959), p. 101

Statistics regarding the geometry exercises

In general, each test contained two exercises on plane geometry, on some occasions just one, and on some occasions three. Yet, in 1931, there was no exercise in plane geometry at all.

Figure 5.
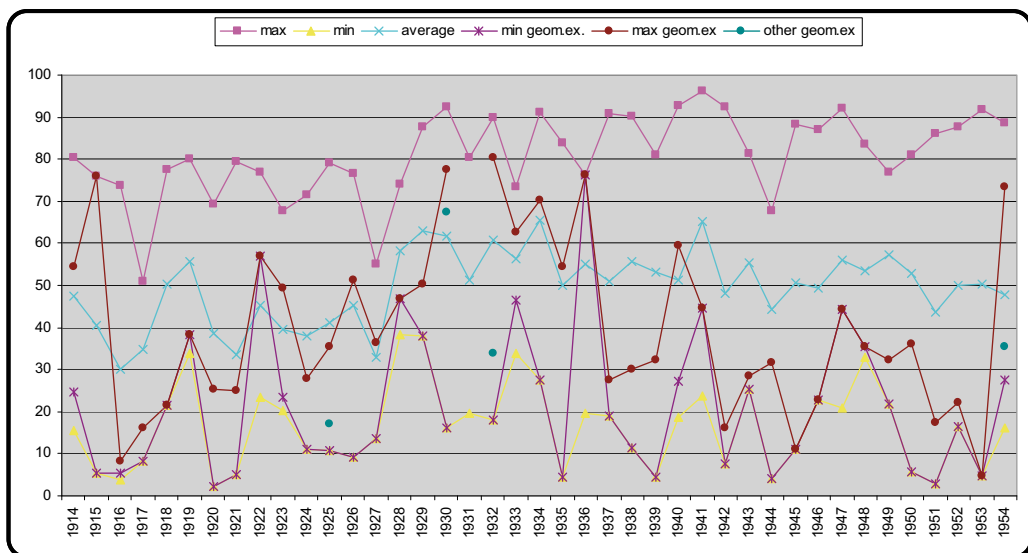
In this diagram, the results at the test in mathematics are related to the number of students that took the test in mathematics. The max-graph indicates the solution frequency for the exercises that was solved by the largest number of students. The min-graph indicates the solution frequency for the exercise that was solved by the smallest number of students. The average-graph indicates the mean value of the solution frequencies of all exercises. The min.geom.ex-graph indicates the solution frequency for the geometry exercises that was solved by the smallest number of students. The max.geom.ex-graph indicates the solution frequency for the geometry exercise that was solved by the largest number of students. The scattered dots indicate the solutions frequency for other geometry exercises. The numbers are based on Riksarkivet A, B and E.

On the basis of the diagram above, it is fair to say that the geometry exercises belonged to the more difficult exercises each year, if not the most difficult. One should then also consider that the share of students that took the test in mathematics dropped from 98 percent in 1935 to 85 percent 1950.

The geometry exercises where the solution frequency is above 50 percent belongs to the Type-I exercises, those that did not require some sort of proof. This type of exercises almost disappeared after 1940. This is a bit surprising considering the fact that the number of lessons was cut by one fifth in 1933. Instead of making the geometry exercises easier, the constructors of the tests made them harder.

If we consider the geometry exercises with the lowest solution frequencies, mainly Type-II exercises, there are no clear trends; the solution frequency could fluctuate quite drastically from year to year. Hence, it is not possible to tell whether the students' performances on the geometry exercises improved or got worse. Fortunately, some of the exam exercises were reused. The test constructors only changed the measurements of angles and lengths.

Table 9.

Exercises	% tries	% success	Exercises	% tries	% success
S 25:6	43.99	35.43	S 39:8	19.46	4.39
S 52:7	24.67	16.60	S 50:8	16.64	5.83
S 39:5	41.21	32.36	S 43:7	66.05	28.46
S 50:6	44.60	36.16	S 52:6	57.85	22.23

Table 9. The shares are related to the number of students taking the exam test in mathematics. V 25:6 is a Type-I exercise. The others are Type-II exercises.

These numbers indicates that the skills of the students had not changed significantly during the 1940's and early 50's. But then we should recall that 90 percent of the students that entered the exam process took the test in mathematics in 1939. In 1950, this figure was 85 percent. Moreover, the test examiners complained about the teachers' corrections of V 50:6 since several teachers had accepted insufficient or even faulty solutions.

Another aspect of diagram O above is that the test constructors seem to have had difficulty in predicting the results and the level of difficulty of the exercises. Some years, the most difficult geometry exercise was solved by 5 percent of the students; other years, by 45 percent of the students. Furthermore, these great differences were not unusual.

Concluding remarks – Geometry exercises in the final examinations and the significance of the professional debate

As I have described in Part C of this thesis, the main goal for the courses in axiomatic geometry was to provide training in reasoning. This goal about training in reasoning was relevant in the sense that the textbooks were designed according to the axiomatic method. As the students, together with the teacher, worked with the textbooks, they were introduced to the whole apparatus of definitions, axioms, theorems, and deductive proofs.

Another aspect of this goal on training in reasoning is that the final exam in mathematics during period 1905-1962 contained at least one geometry

exercise that required that the students be able to provide some sort of justification. At least the exercises were constructed in that way, and according to the exam reports the students were expected to achieve some sort of formal proof.

In comparison to the textbooks, the formulations of the geometry exercises were different. In the exam, the students were not asked to prove some proposition; instead, they were asked to determine an angle, a length, or an area. Moreover, all exercises contained measurements, and almost all solutions required some calculation. But, in order to formulate a numerical or algebraic expression, the students had to discover and establish one to three relations between angles or between lines, for instance, that two angles are equal. These were the relations that should be justified in some way.

Until 1950, at least 85 percent of the students that entered the exam process took the test in mathematics, even though they could choose not to and still pass the exam process. Thus, considering the construction of the geometry exercises on the exam test and the fact that the students were expected to achieve some sort of proof, the goal about training in reasoning encompassed a great majority of the students in Realskolan, not just those who would be attending upper secondary school, i.e. Gymnasiet. In that respect, training in reasoning was a part of a general education.

After 1950, the students could choose between a shorter and a longer course in mathematics. The shorter course did not include axiomatic geometry. After 1950, fewer than 30 percent of the students that entered the exam process took the more difficult exam test in mathematics, the test that covered the courses in axiomatic geometry. In that respect, after 1950 the goal about training did not encompass a majority of the students in Realskolan.[456]

Regarding the requirements on the proofs that the students were supposed to achieve, the proofs in the textbooks constituted the ideal proof. However, the requirements on the proofs performed by the students at the final exams seem to have been lower throughout the period. After 1949, the correction forms established that markings in the diagrams that revealed the student's line of reasoning should pass as a justification. Before 1949, there were no official directives on this issue, but there are comments in the reports about the exams that imply that this was the case. The first comment on this issue appeared in 1925. Moreover, actual student papers and the corrections made by the teachers confirm that this was standard procedure throughout the period 1905-1962.

This puts the relevance of the arguments about training in reasoning and the axiomatic method in a new perspective. In order to pass the geometry exercises on the exam test, the most important thing was not to achieve a textbook proof. The critical part was to discover the relations between angles

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[456] By corresponding schools I am referring to the schools that functioned as experimental schools in connection with the preparations of Grundskolan.

and sides by which a numerical or algebraic expression could be formulated. But, if we consider the debate on geometry instruction in Realskolan, heuristics and problem solving was not an issue. In that respect, the argumentation about training in reasoning did not treat an important aspect of the geometry courses.

The fact that this type of exercises occurred in each exam test and that markings were accepted throughout the period 1905-1962 indicates that proofs were not the teachers' only concern in connection with geometry instruction, especially not with an eye to the final exams.

If we consider the results on the final exams in mathematics, the goal of providing training in reasoning was not a success. Especially, if this type of training were part of a general civic education and not just a preparation for further studies at upper secondary level and university. The results on the more difficult geometry exercises in the final exams were every year among the lowest, if not the lowest. Between 1905 and 1954, half of the geometry exercises that required some sort of justification were solved by fewer than 30 percent of the students that took the test in mathematics. On some occasions, fewer than 10 percent of the students passed these exercises. At this point we should not forget that markings were accepted as justifications. Another circumstance is that the share of students that took the test in mathematics during the exam process decreased from 98 percent in 1935 to 85 percent in 1950. Moreover, from about 1940 and onwards, the number of complaints in the exam reports about students' skills in carrying out formal justifications. Complaints also began to occur about the teachers' abilities to understand and correct the geometry exercises from about this time.

Thus, by the beginning of the 1950's when the preparations for Grundskolan started, the arguments about training in reasoning and general civic education, which I see as the main rationale for having courses in axiomatic geometry, were not especially convincing; they had lost much of their relevance.

Part G – Epilogue

Geometry textbooks and professional literature on geometry instruction in Realskolan, 1905-1962

In the investigated articles regarding geometry instruction in Realskolan, the main goal of geometry instruction was to provide training in reasoning; this was the main argument for having courses in axiomatic geometry. However, the arguments about training in reasoning in connection with geometry instruction included more than logical thinking. A common argument was that the students developed a critical attitude regarding language and reasoning in general, but also their spatial intuition, i.e. their conception of space.

Similar types of arguments were used to justify courses in pure mathematics at the secondary level in Prussia/Germany and England during the 19th century and early 20th century.[457] Still, I do not think that we first and foremost should consider the Swedish debaters during the period 1905-1962 as some kind of defenders of an old educational ideal or some cultural heritage. A more reasonable backdrop to their argumentation is the debate about civil rights during the first decades of the 20th century. I think that the debaters' underlying motivation was to educate independent citizens who are able to think and reason critically. Moreover, if we consider the curricula of Realskolan, we can see that the debaters could support their argumentation on formulations about providing a general civic education. This description is reasonable in comparison to Lövheim's (2006) description of science teachers as a progressive group who wanted to change society.[458] Similar arguments about geometry instruction and civic education also occurred in the USA during the first decades of the 20th century, according to González and Herbst (2006).[459]

But then, apart from the arguments regarding critical thinking, what did training in reasoning mean? What did this training include and what was demanded of the students? One aspect is that all the textbooks up to 1950 were designed according to the axiomatic method. Hence, the students were introduced to the whole apparatus of definitions, axioms, theorems, constructions, and proofs. Not until the end of the 1940's did a textbook with an

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[457] See the background chapters in this thesis.
[458] See the background chapters in this thesis.
[459] See the background chapters in this thesis.

experimental approach appear. But in that case, too, the axiomatic method constituted the core.

Another answer is provided by the final exams in mathematics in Realskolan. Each year, during the period 1905-1962, they contained geometry exercises whose solutions required some sort of justification. More precisely, the students were supposed to justify one to three simple relations regarding angles or lines. This gives us an idea what kind of proofs the students were supposed to master on their own as they left Realskolan. However, in order to pass the geometry exercises the students merely had to make markings in the diagrams or put out measurements that revealed that he or she had recognized the relations. I.e. the students did not have to achieve a more formal justification.

If we then consider also the results of the geometry exercises in the final exams, I think it is fair to say that the goal of providing training in reasoning to a majority of the students in Realskolan was far from attained. Throughout the period 1905-1962, the results of the geometry exercises were quite poor.

Regarding the methodological arguments, spatial intuition was a central issue in the methodological discussions, especially for those who wanted to develop alternatives to Euclid's *Elements* as a textbook. They argued that geometry was more than logical thinking; effective learning required that students also developed their spatial intuition. Moreover, textbooks and teaching should appeal to spatial intuition. Here, we can observe some similarities with how reformists in England and Germany conceived geometry instruction. According to Fujita et al (2006), both Treutlein in Germany and Godfrey in England underscored the importance of spatial intuition in connection with geometry instruction.[460]

A distinctive feature of the debate on spatial intuition is that those who underscored the importance of spatial intuition were linked the teacher journal specialized in mathematics[461], teacher training and the national school board. The same debaters were also careful about marking a difference between school geometry and scientific geometry. Their basic standpoint was that one could not attain the same level of rigor in school geometry.

Another argument in the methodological discussion was that a proof should not only be correct from a logical point of view; it should also reveal a more natural cause to why the theorem is correct. Proofs based on foldings or symmetry were considered more appropriate in that respect. As I have shown in this thesis, textbooks were indeed designed according to these ideas. Moreover, about 1930, these textbooks were used by a majority of the mathematics teachers in the lower secondary schools. To some extent, these findings contradict the claims of Magne (1986) and Håstad (1978) about

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[460] See the background chapters in this thesis.
[461] I.e. *Elementa.*

mathematics instruction being traditional, isolated, and static during the first half of the 20th century.

However, in the articles investigated, Euclid's *Elements* was defended, and it was used by several teachers up to the 1930's. To some extent, Euclid's *Elements* had something of a revival from the 1930's onwards. In 1936, Sjöstedt introduced his textbooks on axiomatic geometry. By his own account, he followed Euclid's course as far as possible. Moreover, he did not insert theorems that included foldings or symmetry. Yet, he deviated from the *Elements* on several points. This textbook, together with an alternative textbook by Olson, were the most common textbooks introduced after 1925, although it is not clear how many teachers actually chose these textbooks.

If we then think of the professional debate as a source of potential arguments and an incentive for actions, it is interesting to consider the relevance of the arguments. What aspects of geometry instruction did the arguments concern? My standpoint is that the arguments were focused on the axiomatic method and proofs. Even though spatial intuition was a central issue in the methodological discussion on how to develop textbooks and teaching, the aim was to develop theorems and proofs. Also when the students' predispositions and their reception of the teaching were discussed, it was the theorems, the proofs, and the axiomatic method that constituted the main concern.

I do not say that these arguments were not relevant. Considering the fact that the teachers could chose between different types of textbooks during the period 1905-1962, they were relevant. My point is that other aspects of geometry instruction, e.g. heuristics or problem-solving, was not really an issue. Moreover, a discussion on problem solving would not have been irrelevant. If we consider the geometry exercises in the final exam and the requirements for the solutions, the most important thing was not to master proofs but to be an able problem solver.

As I see it, an essential part of the geometry courses was not acknowledged as important by the leading debaters. Nor did they provide any notions or methods to tackle problem-solving. In that respect the professional debate was much less relevant and we cannot consider it an incentive for actions and a source of arguments.

This circumstance can be seen as *one* explanation to why the results on the geometry exercises in the final exams were quite bad throughout the period 1905-1962. The lack of explicit notions and methods regarding an essential part of the course may have become critical as an increasing number of new teachers had to be trained. As a matter of fact, from 1940 onwards the reports from the final exams in mathematics indicate that an increasing number of teachers did not grasp the more difficult geometry exercises themselves. Another critical factor was that the number of students in Realskolan increased. But not only that, an increasing number of students probably meant an increasing number of students whose parents had no ex-

perience of the geometry courses in Realskolan. Without support from teachers and parents, the situation of many students was probably quite difficult.

Taken together, I would say that the professional debate about geometry instruction in Realskolan that I have described in this dissertation had lost much of its significance by the late 1940's. The main argument for having courses in axiomatic geometry – the goal about training in reasoning – had lost much of its relevance; a clear majority of the students did not master the geometry exercises on the final exams where they were supposed to apply their skills in reasoning. Moreover, relevant parts of the geometry courses were not covered by the methodological arguments in the professional debate.

I think that this circumstance adds a new perspective to the changes of school geometry that took place in the 1950's and 60's. The professional debate about geometry instruction in Realskolan did not constitute a powerful source of arguments. Arguments by which leading actors, such as Sjöstedt who then held a high position in the national school board, could explain for teachers, school officials, and politicians how and why school geometry should be developed in a certain way. Hence, the changes in mathematics instruction in the 1950's and 60's were not only a matter of arguments about school mathematics being old fashioned in relation to society and science.

To end with, I think it would be interesting to investigate reports and journals connected to the ICMI in a perspective where you consider not only the arguments but also their significance. If we are looking for potential arguments and incentives for actions, I think that ICMI was a leading forum. Not only that, within ICMI we also find prominent mathematicians, like Felix Klein, that may have given some extra weight to certain arguments.

It might be that also the relevance of the reformist arguments conveyed within the ICMI was quite limited. Take, for instance, Treutlein's and Godfrey's views on geometrical thinking, which are described by Fujita et al (2006). Moreover, Godfrey was involved in ICMI. To them, geometrical thinking encompassed logical thinking and abilities linked to spatial intuition, standpoints that look remarkably similar to the arguments in the debate on geometry instruction in Realskolan. [462]

Geometry textbooks and professional literature on geometry instruction in Folkskolan, 1905-1962

If we consider the curricula for Folkskolan during the period 1905-1962, a main goal of geometry instruction was to provide skills that suited the stu-

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[462] See the background chapters in this thesis.

dents' daily life and future working life. In practice, this meant a great many exercises where the students were to calculate lengths, areas, and volumes. All the investigated textbooks of the period were dominated by such exercises. Often these exercises had connotations to some sort of possible future life: the students were supposed to compute areas of fields or volumes of cylindrical pieces of timber.

However, in some of the literature used in teachers training, this goal was complemented by a goal regarding training in reasoning. For instance, this goal was established in the first edition of Wigforss' work on teaching methods in mathematics printed in 1925. This goal was also established in the curriculum of 1955. However, in comparison to the arguments regarding training in reasoning in Realskolan, Wigforss did not restrict this training to geometry.

If we consider the textbooks produced and used during the period 1905-1962, this goal about training in reasoning did not result in large number of new exercises. In the textbooks, exercises where the students were supposed to compute lengths, areas, or volumes dominated. Moreover, the solution to these exercises followed the same procedure: select a formula for length, area, or volume; plug in the given measurements; and compute. Exercises with solutions more complicated than that were very rare throughout the period 1905-1962. In the textbooks first printed before 1925, we find no such exercises. A more difficult exercise could be to determine the measurement of the angle between the hands of a clock at a given time. Thus, if we consider the complexity of the exercises, the goal about training in reasoning appears not to have been relevant to the textbooks authors.

In the investigated essays on geometry instruction in Folkskolan, visualibility (*åskådlighet*) was the important concept in the methodological discussions. Moreover, almost all of the textbook authors designed their textbooks in accordance with the arguments on visualibility.

In all the investigated textbooks intended for Folkskolan, the introductions and explanations of concepts, formulas, and other propositions followed the same routine. The students work through a series of experimental exercises where they are supposed to observe, manipulate, or measure an illustration or some other object. At the end of the series, a definition, a formula, or some other proposition is established. Wigforss and other authors of teaching literature described this routine, and they linked it to the concepts visualibility, i.e. *åskådlighet*. Their main argument was that students, while working with the experimental exercises, should observe the essential features of a concept, a formula, or some other proposition.

Since Wigforss linked his arguments on visualibility and this routine to the goal regarding training in reasoning, it may be that the textbooks authors shared this view. As I have mentioned, the textbook authors were not keen on inserting great numbers of exercises that required more complex solutions.

During the period 1905-1962, the exercises linked to this routine changed. At the beginning of the period, the students were guided through the experimental exercises by written explanations and illustrations. During the 1930's, these written explanations were dropped together with the illustrations. Still, the routine involving experimental exercises was kept intact.

As I see it, the changes in experimental exercises constitute the most thorough change in the geometry textbooks, or the chapters on geometry, during the period 1905-1962. This indicates that the methodological arguments on visualibility were taken seriously by the textbook authors. That was the relevant argument in the development of textbooks. In contrast, we can observe only minor changes of the exercises where the students were supposed to make calculations.

Thus, we can observe how certain aspects of geometry instruction in Folkskolan in fact did change during the period 1905-1962. To some extent, this contradicts Magne's (1986) and Håstad's (1978) claims about mathematics instruction being traditional, isolated, and static during the first half of the 20th century. However, I think the most interesting finding is that the arguments about visualibility appear to have been so influential during the period 1905-1962. In all the textbooks investigated, the experimental routine was applied; a routine that was described by Wigforss and clearly linked to ideas about visualibility. Moreover, Wigforss also linked this routine to arguments about training in reasoning.

If we consider Wigforss' argumentation about mathematics instruction as a source of potential arguments and an incentive for actions, I think we can get some new insights regarding why and how school geometry changed in a certain way in connection with the introduction of Grundskolan. As I have mentioned, Wigforss was very much involved in the preparation of the curriculum for the new Grundskolan.

As a source of potential arguments, Wigforss offered more than advice about how to teach the most basic mathematics useful in the daily life; he also provided methodological arguments and goals about training in reasoning. Moreover, these arguments were relevant to the teachers and others in the sense that they corresponded well to certain parts of the textbooks and the curriculum. We can compare this to the professional debate about geometry instruction in Realskolan. My point is that Wigforss and those who shared his views on mathematics instruction had good opportunities to explain to teachers, school officials, and politicians how and why school mathematics, including geometry, had to change in a certain way.

Summary in Swedish

Syftet med avhandlingen är att undersöka läroböcker och annan litteratur som användes av lärare inom folkskolan och realskolan i samband med undervisningen i geometri under perioden 1905-1962. Källmaterialet utgörs främst av läroböcker i geometri, kursplaner, metodböcker, artiklar som behandlar geometriundervisning och examensprov i matematik för realskolan. Valet av dessa källor motiverar jag med att det är lärare och läroverksförfattare som upprätthåller undervisningen i skolan; om vi då ska söka kunskap om undervisning och hur undervisning förändras, så är det relevant att undersöka texter som denna yrkesgrupp har använt i sin yrkesutövning, sin profession.

Jag har ägnat särskild uppmärksamhet åt något som jag på engelska kallar *professional debate*. På svenska skulle jag vilja kalla det en fackdebatt, en debatt där geometriundervisningens innehåll och metoder beskrivs och motiveras. Alltså, de klassiska didaktiska frågorna om vad, hur och varför.

Min första fråga i avhandlingen är helt enkel: vilkat argument förekom i fackdebatten om den grundläggande geometri undervisningen? De viktigaste källorna för denna fråga har främst varit artiklar i lärartidningar specialiserade på matematikundervisning och litteratur som använts inom lärarutbildningen i samband med matematik.

Den andra frågan handlar om fackdebattens signifikans, en fråga som delas upp på två delfrågor.

a) Vad betydde de begrepp och argument som förekom i debatten? Ett vanligt argument var att undervisningen skulle vila på åskådlighet, men vad betyder egentligen det? Ett annat uttryck man kan ställa sig undrande inför är "träning i tänkande". Särskilt den axiomatiska geometrin på realskolan sågs som viktig i detta avseende. Men återigen, vad innebär denna träning egentligen? Vad krävdes av eleverna? Detta försöker jag att utreda.

b) Den andra delfrågan handlar om argumentens relevans. I vilken utsträckning berörde argumenten i fackdebatten kursernas innehåll och de professionellas verksamhet?

För att besvara frågan om fackdebatens signifikans har jag främst undersökt läroböcker och examensprov.

Skälet till att undersöka den grundläggande geometriundervisningen i Sverige under perioden 1905-1962 är tvåfaldigt. För det första, skolgeometrin diskuterades flitigt i andra länder under denna period. Vad som skedde i Sverige har vi dock liten kännedom om, även om geometri var en

viktigt del i skolmatematiken i Sverige. För det andra, det svenska skolsystemet genomgår under perioden 1905-1962 en rad viktiga förändringar: naturvetenskaperna blir viktiga skolämnen; antalet elever på realskolan och gymnasiet ökar; kvinnor får tillgång till högre utbildning; folkskolan och realskolan integreras; och kyrkans inflytande över skolan minskas. Eftersom matematik var, och fortfarande är, ett viktigt skolämne, så är en undersöksökning av detta ämne under perioden 1905-1962 intressant också ur ett mer allmänt utbildningshistoriskt perspektiv. Inte minst är skolgeometrin intressant då den upptog en stor del i timplanen.

Som sagt, den svenska skolmatematikens historia är relativt okänd. Antalet mer genomgripande studier kursplaner, läroböcker, artiklar, prov, mm, är mycket litet. Vad jag vet finns det en sådan studie, Hatami (2007). Det det finns dock i didaktiska och pedagogiska rapporter och avhandlingar historiska bakgrunder där skolmatematiken historia i Sverige berörs. De få gånger som perioden före 1950 berörs, så görs det i termer av "tradition", "statisk" och "isolation".

Vid en jämförelse med arbeten kring Sveriges utbildningshistoria, så intar dessa beskrivningar något av en särställning. Ofta beskrivs perioden 1905-1962 som en period då naturvetenskap, ingenjörskonst, teknologi, upplysning, demokrati och jämlikhet var viktiga begrepp i debatten om hur samhällets skulle förändras. Inte minst deltog en del läroverkslärare i naturvetenskap i denna debatt. Om beskrivningarna om "tradition", "statisk" och "isolation" är riktiga, så antyder det att skolmatematikens innehåll inte följde samhällsdebatten, en debatt som en del lärare själva deltog i.

I detta sammanhang kan geometriundervisningen på realskolan framstå som något som hörde till det traditionella. De senare geometrikurserna på realskolan var fokuserade på den axiomatiska geometrin och bevis och tilllämpningar togs inte upp i särskilt stor utsträckning.

För att slutningen nämna något om mina resultat, så visar jag att geometriundervisningen i Sverige förändrades under perioden 1905-1962. I den fackdebatt som rörde geometriundervisningen på realskolan, så diskuterades läroböcker och undervisningsmetoder. Kritik framfördes mot läroböcker som följde Euklides *Elementa* och man beskrev hur alternativa läroböcker borde vara utformad. Men inte bara det, man producerade också läroböcker i enlighet med dessa förslag. Vid början av 1930-talet var dessa böcker betydligt mer populära än de böcker som följde Euklides *Elementa*. I de nya böckerna introducerade man bl a satser som byggde på symmetribegreppet och vikningar. Sådana satser motiverades med att de var mer naturliga än de gamla Euklidiska. En viktig detalj är dock att den axiomatiska metoden inte övergavs. Den skulle göras mer tillgänglig med hjälp av dessa nya satser och bevis.

Även läroböckerna på folkskolan förändrades. I fackdebatten om matematikundervisningen i Folkskolan var det viktigt att begrepp och formler presenterades på ett åskådligt vis. En väsentlig del i ett sådant arbetssätt var att

eleverna var aktiva genom att observera och manipulera olika objekt eller figurer. Detta arbetssätt genomsyrande även utformningen av läroböckerna i geometri. Genom obsertvationer och experiment introducerades geometriska begrepp och formler för eleverna. Under perioden 1905-1962 kan men se förändring av hur också folkskolans läroböcker i geometri utformades med avseende på detta arbetssätt. I de böcker som producerats före 1925, så guidas eleverna igenom åskådningsövningarna. Efter 1925 är en del läroboksförfattare betydligt sparsammare med hjälpen och eleverna ska klara uppgifterna på egen hand. Men i övrigt skedde relativt få förändringar av folkskolans läroböcker i geometri.

Även inom realskolan fanns det saker som inte förändrades. Under hela perioden 1905-1962, så motiverades kurserna i axiomatisk geometri med att de erbjöd en utmärkt träning i tänkande. Det var dock inte bara en fråga om träning i logik, geometriundervisningen ansågs också bidra till att utveckla ett allmänt kritiskt förhållningssätt. En annan aspekt som inte heller förändras i fackdebatten kring realskolans geometriundervisning var att diskussionerna var fokuserade på den axiomatiska metoden; hur kunde bevisen utvecklas, hur kunde man få eleverna att förstå värdet av bevis, mm. Andra frågor, så exempelvis problemlösning diskuterades mycket litet. En sådan diskussion hade dock inte varit irrelevant. Om man ser till examensproven i matematik vid realskolan, så var det viktigare att kunna lösa problem än att kunna presetera bevis enligt läroboken för att lyckas på provens geometriuppgifter.

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Appendixes

A – Prop.I.5 in Euclid's *Elements*

Heath's edition. Heath (1956), pp. 251-252

In isosceles triangles the angles at the base are equal to one another, and, if the equal straight lines be produced further, then the angles under the base will be equal to one another.

Let *ABC* be an isosceles triangle having the side *AB* equal to the side *AC,* and let the straight lines *BD*, *CE* be produced further in a straight line with *AB*, *AC.* (Post. 2)
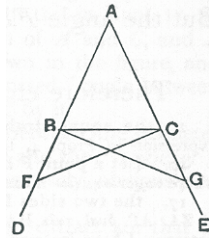
I say that the angle *ABC* is equal to the angle *ACB,* and the angle *CBD* to the angle *BCE.*

Let a point *F* be taken at random on *BD*; from *AE* the greater let *AG* be cut off equal to *AF* the less (I.3); and let straight lines *FC*, *GB* be joined. (Post. 3)

Then, since *AF* is equal to AG and *AB* to *AC,* the two sides *FA*, *AC* are equal to the two sides *GA*, *AB,* respectively; and they contain a common angle, the angle *FAG.*

Therefore the base *FC* is equal to the base *GB,* and the triangle *AFC* is equal to the triangle *AGB,* and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend, that is, the angle *ACF* to the angle *ABG,* and the angle *AFC* to the angle *AGB.* (I.4)

And, since the whole *AF* is equal to the whole *AG,* and in these *AB* is equal to *AC,* the remainder *BF* is equal to the remainder *CG.*

But *FC* was also proved equal to *GB*; therefore the two sides *BF*, *FC* are equal to the two sides *CG*, *GB* respectively; and the angle *BFC* is equal to the angle *CGB,* while the base *BC* is common to them; therefore the triangle *BFC* is also equal to the triangle *CGB,* and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend; therefore the angle *FBC* is equal to the angle *GCB,* and the angle *BCF* to the angle *CBG.*

Accordingly, since the whole angle *ABG* was proved equal to the angle *ACF,* and in these the angle *CBG* equals the angle *BCF,* the remaining angle *ABC* equals the remaining angle *ACB,* and they are at the base of the triangle *ABC.* But the angle *FBC* was also proved equal to the angle *GCB,* and they are under the base.

Therefore etc.   Q.E.D.

B – Olson's list of axioms

Olson (1940), p. 2
1. The whole is larger than each of its parts but equal to the sum of the parts.
2. Those which are equal to one and the same are mutually equal.
3. That which is larger then one of two equals is also larger than the other.
4. That which is smaller than one of two equals is also smaller than the other.
5. If equals are added to equals, the sums are equal.
6. If equals are reduced by equals, the differences are equal.
7. The same multiples or parts of equals are equal.
8. If one quantity is larger than another and the quantities are increased (decreased) by equals, then the first sum (difference) is larger than the other.
9. Hereby we assume as self-evident that lines and figures that we imagine to change position in space thereby do not undergo any kind of change with respect to shape and size.
10. Through a point, only one straight line can be drawn parallel to a given straight line.

C – Sjöstedt's list of axioms

Sjöstedt (1936), pp. 12-14
1. Through two points, only one straight line can run.
2. Through a point outside a straight line, only one straight line can run parallel to the latter.
3. If a distance in a figure changes size from one value to another, then it runs through every value between these.[464]
4. Quantities that are equally great can replace each other.
5. The whole is greater than each of its parts.
6. If one quantity is greater than another that is greater than a third, then the first is greater than the third.
7. If equals are increased by equal, the sums are equal.
8. If equals are decreased by equal, the differences are equal.
9. If one quantity is greater than another and both are increased by equals, then the former sum is greater than the latter.
10. If one quantity is greater than another and both are decreased by equals, then the former difference is greater than the latter.
11. Quantities that can cover each other are equal in size.

D – List of theorems in the first chapter of Asperén's textbook

The numbing of Euclid's propositions is taken from Heath's edition.

Table 10.

Propositions, Asperén	Propositions, Euclid
Th.II.c	I.13
Th.II.d	I.15
Th.III.a	-
Th.III.b	-
Prob.4	I.12
Prob.5	I.11
Prob.6	I.10
Prop.7	I.9
Prop.8	I.23
Th.IV.a	-
Th.IV.b	-
Th.IV.c	-
Th.IV.d	-
Th.V.a	I.27
Prob.9	I.31
Axiom of parallels	
Th.V.b	Post.5
Th.V.c	I.29
Th.VI	I.20

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[464] Sjöstedt (1936), pp. 12-13

Prob.10	I.22
Th.VII.a-b	I.32
Th.VIII.a	I.5
Th.VIII.b	I.18
Th.VIII.c	I.6
Th.VIII.d	I.19
Prob.11	-
Th.IX.a	I.4
Th.IX.b	I.24
Th.X.a	I.8
Th.X.b	I.25
Prob.12	-
Th.XI	I.26
Prob.13	-
Th.XII	-

E – Outline of the first chapter of Sjöstedt's textbook

The numbing of Euclid's propositions is taken from Heath's edition.

Table 11.

Propositions, Sjöstedt	Propositions, Euclid
Axioms	
I. Straight lined diagrams	
1. Angles	
1	I.13
2	I.15
2. The congruence of triangles	
3	I.4
4	-
5	I.5
6	I.6
7	I.8
8	I.22
3. Some basic constructions	
14	-
15	I.27
16	I.31
17	I.28
18	I.28
19	I.29
20	Alt. I.29
21	Alt. I.29
5. The sum of angles in a triangle	

22	I.32
23	I.32
24	I.26
6. Sides and angles in triangles	
25	I.25
26	I.24
27	I.20
28	-
29	-
30	I.24
31	I.25
32	-
7. Parellelograms	
33	I.34
34	-
35	-
8. Figures with equal surfaces	
36	I.35
37	I.41
38	I.47

F – Nyhlén's list of axioms

Nyhlén (1947), pp. 2-18

1. Through two points there can run only one straight line.
2. If *AB* is a distance and *A´* is a point on a straight line, then you can determine exactly one point *B´*, on each side of *A´*, so that *A´B´* equals *AB*.
3. If two distances equals a third, then they are mutually equal.
4. If two distances equal two other distances respectively, then the sum of the former equals the sum of the latter.
5. If *AOB* is an angle and *O´A´* is a ray, then you can determine exactly one ray *O´B´* on each side of the line *O´A´*, so the $\wedge A´O´B´$ equals $\wedge AOB$.
6. If two angles equals a third, then they are mutually equal.
7. If two angles equal two other angles respectively, then the sum of the former equals the sum of the latter.
8. Construct an *Δ ABC*. Allocate $\wedge A´ = \wedge A$ and on the legs the points *B´* and *C´* so that *A´B´ = AB* and *A´C´ = AC*. Draw *B´C´*. Compare *BC* and *B'C'*, $\wedge B$ and $\wedge B´$ and $\wedge C$ and $\wedge C´$ by means of transporters.
9. Parallel lines generate equal alternate angles with each transversal.